International Journal of General Systems

ISSN: 0308-1079 (Print) 1563-5104 (Online) Journal homepage: http://www.tandfonline.com/loi/ggen20

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INTERNATIONAL JOURNAL OF GENERAL SYSTEMS, 2018
https://doi.org/10.1080/03081079.2018.1445738

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ABSTRACT

In a recent paper, we have introduced the notion of clone relation of a given binary relation. Intuitively, two elements are said to be "clones" if they are related in the same way w.r.t. every other element. In this paper, we generalize this notion from pairs of elements to sets of elements of any cardinality, resulting in the introduction of clonal sets. We investigate the most important properties of clonal sets, paying particular attention to the introduction of the clonal closure operator, to the analysis of the (lattice) structure of the set of clonal sets and to a distance metric expressing how close two elements are to being clones.

ARTICLE HISTORY
Received 10 November 2017
Accepted 12 February 2018

KEYWORDS
Clone relation; clonal set; clonal closure; clonal lattice; clonal distance metric

1. Introduction

The study of binary relations probably is one of the oldest study subjects in modern mathematics (Peirce 1880). Of particular interest are tolerance and equivalence relations, which model whether two elements are "alike" or not. This premise of grouping together elements that are "alike" represents the core of many scientific disciplines, such as cluster analysis (Dave 1990; Everitt et al. 2011) and formal concept analysis (Ganter and Wille 1999; Bělohlávek and Vychodil 2012).

In this same direction, De Baets, Zedam, and Kheniche (2016) introduced the clone relation of a given strict order relation for analysing whether two elements behave in the same manner within the given strict order relation. Later on, Bouremel et al. (2017) generalized the definition of clone relation to any type of binary relation by calling two elements clones if they are related in the same way w.r.t. every other element. The resulting relation turned out to be an interesting study subject that, despite what one could think at first glance, does not need to be an equivalence relation in general for non-symmetric relations. This clone relation has proved to have applications in the study of compatibility of binary fuzzy relations (Höhle and Blanchard 1985; Bělohlávek 2004; Kheniche, De Baets, and Zedam 2015). More specifically, the clone relation has been used for studying the compatibility of a crisp relation with a fuzzy equivalence relation (De Baets, Bouremel, and

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Zedam 2016) and for characterizing the fuzzy tolerance and fuzzy equivalence relations that a strict order relation is compatible with (De Baets, Zedam, and Kheniche 2016).

Independently and just for the case of rankings (strict total order relations), the notion of clones has been studied for decades in a totally different field[1]: social choice theory (Tideman 1987; Zavist and Tideman 1989). In this context, two elements are said to be clones if they are consecutive in the given ranking. Note that this definition coincides with the definition by Bouremel et al. (2017) when restricted to rankings. Due to the key role played by clones in social choice theory, where they can completely change the outcome of an election procedure, a vast part of the literature in social choice theory has focused on the search for methods that are independent of clones, with ranked pairs (Tideman 1987) and the method of Schulze (2011) being two of the most prominent examples. However, in this field, the notion of clones is not only restricted to pairs of elements, but is applied to any possible set of cardinality greater than or equal to two. The considered rationale is still the same though: a set of elements is said to be a set of clones if all elements in the set are related in the same way w.r.t. every element not belonging to that set. In this paper, we propose to follow this direction and introduce the notion of clonal set of a binary relation. Many interesting results will follow. Here, we highlight the introduction of the clonal closure operator, the complete lattice structure of the set of clonal sets, and a natural distance metric measuring the clonal distance between two elements.

The remainder of the paper is structured as follows. In Section 2, we introduce the notion of clonal set and discuss the most important properties. In Section 3, the clonal closure operator is introduced and used to prove that the set of clonal sets is a complete lattice. In Section 4, we introduce a distance metric that measures how close two elements are to being clones. In Section 5, the particular cases in which the given relation is an equivalence relation or a weak order relation are analysed. We end with some conclusions in Section 6.

2. Clonal sets

2.1. The clone relation of a binary relation

A (binary) relation on a set X is a subset of X^2, i.e. a set of couples $(x, y) \in X^2$. For a relation R on X, we often write xRy instead of $(x, y) \in R$. We denote by R^c the complement of the relation R, i.e. for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R, i.e. for any $x, y \in X$, xR^ty denotes the fact that yRx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. The set difference of two relations R and S on a set X is the relation $R\backslash S$ on X defined as $R\backslash S = \{(x, y) \in X^2 \mid xRy \wedge xS^cy\}$. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$.

We recall that common properties of a relation R on a set X are: reflexivity (xRx, for any $x \in X$); irreflexivity (xR^cx, for any $x \in X$); symmetry (xRy implies that yRx, for any $x, y \in X$); antisymmetry (($xRy \wedge yRx$) implies that $x = y$, for any $x, y \in X$); transitivity (($xRy \wedge yRz$) implies that xRz, for any $x, y, z \in X$); and completeness (either xRy or yRx

$$R = \begin{array}{c} \\ x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_1 & \dots & x_i & & \dots & & x_j & \dots & x_n \end{array} \\ \begin{pmatrix} R_{11} & \dots & R_{1i} & \dots & R_{1j} & \dots & R_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{i1} & \dots & R_{ii} & \dots & R_{ij} & \dots & R_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{j1} & \dots & R_{ji} & \dots & R_{jj} & \dots & R_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{n1} & \dots & R_{ni} & \dots & R_{nj} & \dots & R_{nn} \end{pmatrix} \end{array}$$

Figure 1. Natural interpretation of the clone relation based on the matrix representation of R.

holds, for any $x, y \in X$). A relation $\leq$ on a set X is called an order relation if it is reflexive, antisymmetric and transitive. If an order relation is complete, then one talks about a total order relation. A set X equipped with an order relation $\leq$ is called a partially ordered set (poset, for short), denoted by $(X, \leq)$. For more details on (binary) relations, we refer to Davey and Priestley (2002); Schröder (2002).

In the following, we recall the notion of clone relation of a (binary) relation introduced by Bouremel et al. (2017), which generalizes the notion of clone relation of a strict order relation introduced by De Baets, Zedam, and Kheniche (2016). Informally, two elements are said to be clones (with respect to a given relation) if they are related in the same way with any other element.

Definition 2.1 (Bouremel et al. 2017): Let R be a relation on a set X. The clone relation $\approx_R$ of R is the relation on X defined by

$$x \approx_R y \quad \text{if} \quad \begin{cases} (\forall z \in X \setminus \{x, y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x, y\})(xRz \Leftrightarrow yRz)\,. \end{cases}$$

If $x \approx_R y$, then we say that x and y are clones w.r.t. the relation R. We recall that the clone relation $\approx_R$ of a relation R on X is a tolerance relation on X, i.e. a reflexive and symmetric relation.

The matrix representation of a relation R can be used for illustrating the notion of clone relation and for facilitating the identification of clones in the finite case. Let R be a relation on a finite set $X = \{x_1, x_2, \dots, x_n\}$. For any $x_i, x_j \in X$ with $1 \leq i, j \leq n$, it holds that[2]

$$R_{ij} = \begin{cases} 1, & \text{if } x_i R x_j\,, \\ 0, & \text{if } x_i R^c x_j\,. \end{cases}$$

By definition, it holds that $x_i \approx_R x_j$ if and only if, for any $k \notin \{i, j\}$, $R_{ik} = R_{jk}$ and $R_{ki} = R_{kj}$. This means that x_i and x_j are clones if and only if the row and column corresponding to x_i coincide with the row and column corresponding to x_j, with the exception of the four elements contained in the intersection of these two rows with these two columns. This is illustrated in Figure 1.

Example 2.2: Consider the symmetric relations R_D and R_B associated with the diamond graph and the butterfly graph represented in Figure 2.

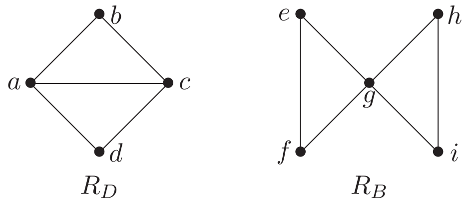

Figure 2. Diamond graph (left) and butterfly graph (right).

For instance, for R_D we can see that a and c are related in the same way to b and d, thus, a and c are clones. The clone relation of R_D and the clone relation of R_B are given by:

$$\approx_{R_D} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \approx_{R_B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

2.2. Clonal sets of a binary relation

The notion of clone relation can be naturally extended to more than two elements, resulting in the introduction of clonal sets. Informally, a clonal set is a set of which any two elements are related in the same way with any other element not belonging to this set.

Definition 2.3: Let R be a relation on a set X. A subset A of X is called a clonal set of R if

$$(\forall x, y \in A)(\forall z \in X \setminus A)((zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)).$$

We denote by $\mathcal{C}_R$ the set of all clonal sets of a relation R. Obviously, if $|X| \leq 2$, then it holds that $\mathcal{C}_R = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.

By definition, the fact that two elements of a subset are related (or not) has no impact on this subset being a clonal set. In particular, the reflexivity of the given relation has no impact on a subset being a clonal set.

Proposition 2.4: *Let R_1 and R_2 be two relations on a set X and A be a subset of X. If $R_1 \backslash A^2 = R_2 \backslash A^2$, then it holds that $A \in \mathcal{C}_{R_1}$ if and only if $A \in \mathcal{C}_{R_2}$.*

As before, the matrix representation of a relation can be used for illustrating the notion of clonal set in the finite case. Let R be a relation on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ and A be a subset of X. We denote by I_A the set of indices corresponding to A, i.e. $I_A = \{i \in \{1, 2, \ldots, n\} \mid x_i \in A\}$. By definition, A is a clonal set of R if and only if, for any $i, j \in I_A$ and any $k \notin I_A$, it holds that $R_{ik} = R_{jk}$ and $R_{ki} = R_{kj}$. This means that A is a clonal set of R if and only if the row and column corresponding to any element $x_i \in A$ coincide with the row and column corresponding to any other element $x_j \in A$, with the exception of the $|A|^2$ elements contained in the intersection of these $|A|$ rows with these $|A|$ columns. This is illustrated in Figure 3.

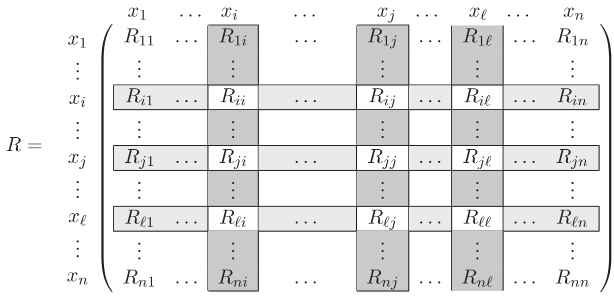

Figure 3. Natural interpretation of the clonal set $A = \{x_i, x_j, x_\ell\}$ based on the matrix representation of R.

Example 2.5: Consider again the relations R_D and R_B given in Example 2.2. The set of clonal sets of R_D and the set of clonal sets of R_B are given by:

$$\mathcal{C}_{R_D} = \left\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,c\}, \{b,d\}, \{a,b,c,d\}\right\},$$
$$\mathcal{C}_{R_B} = \left\{\emptyset, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{e,f\}, \{h,i\}, \{e,f,h,i\}, \{e,f,g,h,i\}\right\}.$$

Note that some particular subsets can be easily identified as clonal sets.

Proposition 2.6: *Let R be a relation on a set X and A be a subset of X.*

(i) *If $A = \emptyset$, then $A \in \mathcal{C}_R$. Therefore, $\emptyset$ is the smallest clonal set of R.*
(ii) *If A is a singleton, then $A \in \mathcal{C}_R$.*
(iii) *If A consists of two elements, say $A = \{x, y\}$, then $A \in \mathcal{C}_R$ if and only if $x \approx_R y$.*
(iv) *If $A = X$, then $A \in \mathcal{C}_R$. Therefore, X is the largest clonal set of R.*
(v) *For any element $a \in X$, it holds that the tolerance class $[a]_{\approx_R} = \{b \in X \mid b \approx_R a\}$ is a clonal set of R.*

The set of clonal sets of a given relation always coincides with the set of clonal sets of its complement and its transpose.

Proposition 2.7: *Let R be a relation on a set X. Then it holds that $\mathcal{C}_R = \mathcal{C}_{R^c} = \mathcal{C}_{R^t}$.*

***Proof*:** First, we prove that $\mathcal{C}_R = \mathcal{C}_{R^c}$. For any $A \in \mathcal{C}_R$, it holds that

$$\begin{aligned} A \in \mathcal{C}_R &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)) \\ &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((zR^c x \Leftrightarrow zR^c y) \wedge (xR^c z \Leftrightarrow yR^c z)) \\ &\Leftrightarrow A \in \mathcal{C}_{R^c}. \end{aligned}$$

Similarly, we prove that $\mathcal{C}_R = \mathcal{C}_{R^t}$. For any $A \in \mathcal{C}_R$, it holds that

$$\begin{aligned} A \in \mathcal{C}_R &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((zRx \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow yRz)) \\ &\Leftrightarrow (\forall x, y \in A)(\forall z \in X \setminus A)((xR^t z \Leftrightarrow yR^t z) \wedge (zR^t x \Leftrightarrow zR^t y)) \\ &\Leftrightarrow A \in \mathcal{C}_{R^t}. \end{aligned}$$

□

Remark 1: If A is a clonal set of a relation R on a set X, then A^c does not necessarily need to be a clonal set of R, as can be seen in the following example. Consider the set

$X = \{a, b, c\}$ and the relation $R = \{(a, c), (b, c)\}$. Since $A = \{a\}$ is a singleton, it holds that $A \in \mathcal{C}_R$, while one can easily verify that $A^c = \{b, c\} \notin \mathcal{C}_R$.

Next, we discuss the intersection and union of clonal sets. First, we prove that any family of clonal sets is closed under intersection.

Proposition 2.8: *Let R be a relation on a set X. For any family $(A_i)_{i\in I}$ of clonal sets of R, it holds that $\cap_{i\in I} A_i \in \mathcal{C}_R$.*

***Proof*:** Let $x, y \in \cap_{i\in I} A_i$ and $z \in X \setminus \cap_{i\in I} A_i$. There exists $i_0 \in I$ such that $x, y \in A_{i_0}$ and $z \in X \setminus A_{i_0}$. Since $A_{i_0} \in \mathcal{C}_R$, it follows that $zRx \Leftrightarrow zRy$ and $xRz \Leftrightarrow yRz$. Hence, it holds that $\cap_{i\in I} A_i \in \mathcal{C}_R$. □

In general, the union of a family of clonal sets does not need to be a clonal set, as can be seen in Example 2.5. For instance, $\{a\}, \{b\} \in \mathcal{C}_{R_D}$, while $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{C}_{R_D}$. However, in case their intersection is not empty, the union of a family of clonal sets is assured to be a clonal set.

Proposition 2.9: *Let R be a relation on a set X. For any family $(A_i)_{i\in I}$ of clonal sets of R, if $\cap_{i\in I} A_i \neq \emptyset$, then it holds that $\cup_{i\in I} A_i \in \mathcal{C}_R$.*

***Proof*:** Let $x, y \in \cup_{i\in I} A_i$ and $z \in X \setminus \cup_{i\in I} A_i$. Hence, $z \in X \setminus A_i$, for any $i \in I$, and there exist $j, k \in I$ such that $x \in A_j$ and $y \in A_k$. Since $\cap_{i\in I} A_i \neq \emptyset$, it follows that $A_j \cap A_k \neq \emptyset$, which implies that there exists t such that $t \in A_j$ and $t \in A_k$. Since $A_j, A_k \in \mathcal{C}_R$, $x, t \in A_j$ and $y, t \in A_k$, it holds that

$$(zRx \Leftrightarrow zRt \Leftrightarrow zRy) \wedge (xRz \Leftrightarrow tRz \Leftrightarrow yRz)\,.$$

This implies that $zRx \Leftrightarrow zRy$ and $xRz \Leftrightarrow yRz$. Hence, $\cup_{i\in I} A_i \in \mathcal{C}_R$. □

Corollary 2.10: *Let R be a relation on a set X. For any $x, y, z \in X$ such that $x \approx_R y$ and $y \approx_R z$, it holds that $\{x, y, z\} \in \mathcal{C}_R$.*

For any $n \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$, the nth power R^n of a relation R on X is recursively defined as:

$$(R^1 = R) \text{ and } (\forall n \in \mathbb{N}^*)(R^{n+1} = R^n \circ R)\,.$$

Let R^* denote the transitive closure of a relation R on a set X, i.e. the smallest transitive relation on X that contains R. The transitive closure R^* can be characterized as:

$$R^* = \bigcup_{k\geq 1} R^k\,.$$

The transitive closure of a reflexive (resp. symmetric) relation is reflexive (resp. symmetric) as well. For more details on transitive closures, we refer to Lidl and Pilz (1998).

Proposition 2.11: *Let R be a relation on a set X and $n \in \mathbb{N}^*$. It holds that*

$$\mathcal{C}_{\bigcup_{i=1}^{n} R^i} \subseteq \mathcal{C}_{\bigcup_{i=1}^{n+1} R^i}\,.$$

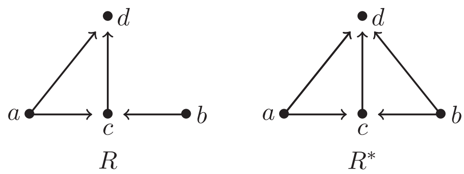

Figure 4. Graph of the relation R (left) and of its transitive closure R^* (right) in Example 2.13.

Proof: Let us denote $R_n := \bigcup_{i=1}^{n} R^i$ and $R_{n+1} := \bigcup_{i=1}^{n+1} R^i$. Let $A \in \mathcal{C}_{R_n}$, then we need to prove that $A \in \mathcal{C}_{R_{n+1}}$. Consider $x, y \in A$ and $z \in X \setminus A$ such that $zR_{n+1}x$ (the case in which $xR_{n+1}z$ is proved analogously). It follows that zR_nx or $zR^{n+1}x$.

(a) If zR_nx, then, as $A \in \mathcal{C}_{R_n}$, $x, y \in A$ and $z \in X \setminus A$, it follows that zR_ny. Hence, $zR_{n+1}y$.
(b) If $zR^{n+1}x$, then there exists $t \in X$ such that zRt and tR^nx. This implies that zR_nt and tR_nx. We distinguish two cases: $t \in A$ or $t \notin A$.
   (α) If $t \in A$, then, as $A \in \mathcal{C}_{R_n}$, $t, y \in A$, $z \in X \setminus A$ and zR_nt, it follows that zR_ny. Hence, $zR_{n+1}y$.
   (β) If $t \notin A$, then, as $A \in \mathcal{C}_{R_n}$, $x, y \in A$, $t \in X \setminus A$ and tR_nx, it follows that tR_ny. Since zRt and tR_ny, it follows that $zR_{n+1}y$. Hence, $zR_{n+1}y$.

We conclude that $\mathcal{C}_{R_n} \subseteq \mathcal{C}_{R_{n+1}}$. □

Corollary 2.12: *Let R be a relation on a set X and R^* be its transitive closure. It holds that $\mathcal{C}_R \subseteq \mathcal{C}_{R^*}$.*

Note that the converse of the above corollary does not necessarily hold, as can be seen in the following example.

Example 2.13: Let R be the relation on $X = \{a, b, c, d\}$ defined as $R = \{(a, c), (a, d), (b, c), (c, d)\}$. One easily verifies that $R^* = \{(a, c), (a, d), (b, c), (b, d), (c, d)\}$. It holds that $\{a, b, c\} \in \mathcal{C}_{R^*}$, while $\{a, b, c\} \notin \mathcal{C}_R$ (Figure 4).

Finally, we show that the intersection of the sets of clonal sets of two relations can be expressed in terms of the clonal sets of the intersection, union and set difference of both relations.

Proposition 2.14: *Let R and S be two relations on a set X. The following statements hold:*

(i) $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$;
(ii) $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R\cup S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$.

Proof:

(i) We need to prove that $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$ and that $\mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R} \subseteq \mathcal{C}_R \cap \mathcal{C}_S$.
   (a) Let A be a subset of X such that $A \in \mathcal{C}_R \cap \mathcal{C}_S$. For any $x, y \in A$ and for any $z \in X \setminus A$, it holds that

$$\begin{aligned} z(R\cap S)x &\Leftrightarrow (zRx \wedge zSx) \\ &\Leftrightarrow (zRy \wedge zSy) \\ &\Leftrightarrow z(R\cap S)y\,. \end{aligned}$$

Similarly, it holds that

$$x(R\cap S)z \;\Leftrightarrow\; y(R\cap S)z\,.$$

Hence, $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\cap S}$.
Moreover, for any $z \in X \setminus A$ and for any $x, y \in A$, it holds that

$$\begin{aligned} z(R\setminus S)x &\Leftrightarrow (zRx \wedge zS^c x)\\ &\Leftrightarrow (zRy \wedge zS^c y)\\ &\Leftrightarrow z(R\setminus S)y\,. \end{aligned}$$

Similarly, it holds that

$$x(R\setminus S)z \;\Leftrightarrow\; y(R\setminus S)z\,.$$

Hence, $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{R\setminus S}$. The fact that $\mathcal{C}_R \cap \mathcal{C}_S \subseteq \mathcal{C}_{S\setminus R}$ is proved analogously.

(b) Let A be a subset of X such that $A \in \mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$. For any $x, y \in A$ and for any $z \in X \setminus A$, it holds that

$$\begin{aligned} zRx &\Leftrightarrow zRx \wedge (zSx \vee zS^c x)\\ &\Leftrightarrow (z(R\cap S)x) \vee (z(R\setminus S)x)\\ &\Leftrightarrow (z(R\cap S)y) \vee (z(R\setminus S)y)\\ &\Leftrightarrow zRy\,. \end{aligned}$$

Similarly, it holds that

$$xRz \;\Leftrightarrow\; yRz\,.$$

Hence, $\mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R} \subseteq \mathcal{C}_R$.
Similarly, it also holds that $\mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R} \subseteq \mathcal{C}_S$.

Hence, $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R\cap S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$.

(ii) From (i), it follows that $\mathcal{C}_{R^c} \cap \mathcal{C}_{S^c} = \mathcal{C}_{R^c\cap S^c} \cap \mathcal{C}_{R^c\setminus S^c} \cap \mathcal{C}_{S^c\setminus R^c}$. Since $\mathcal{C}_{R^c} = \mathcal{C}_R$, $\mathcal{C}_{S^c} = \mathcal{C}_S$, $\mathcal{C}_{R^c\cap S^c} = \mathcal{C}_{(R\cup S)^c} = \mathcal{C}_{R\cup S}$ (see Proposition 2.7), $R^c \setminus S^c = S \setminus R$ and $S^c \setminus R^c = R \setminus S$, it follows that $\mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_{R\cup S} \cap \mathcal{C}_{R\setminus S} \cap \mathcal{C}_{S\setminus R}$. □

3. The lattice structure of the set of clonal sets

3.1. The clonal closure operator

The notion of closure operator is a fundamental notion in mathematics (Moore, 1910). Formally, a closure operator on a set X is a mapping $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying extensivity ($A \subseteq \mathrm{cl}(A)$, for any $A \in \mathcal{P}(X)$), monotonicity ($A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, for any $A, B \in \mathcal{P}(X)$) and idempotency ($\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$, for any $A \in \mathcal{P}(X)$). A subset A of X is called closed if $\mathrm{cl}(A) = A$. A related notion is that of a closure system, a subset $\mathcal{E}$ of $\mathcal{P}(X)$ that is closed under arbitrary intersections ($(A_i)_{i\in I} \in \mathcal{E}$ implies $\cap_{i\in I} A_i \in \mathcal{E}$) and contains X. Propositions 2.6 and 2.8 imply that the set of clonal sets of a given relation is a closure system.

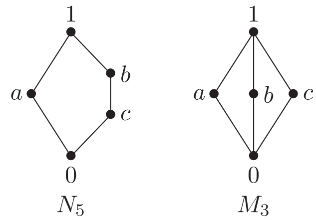

Figure 5. Hasse diagram of the lattice N_5 (left) and M_3 (right).

It is well known that a closure system defines a closure operator and vice-versa (see, e.g. Casparda and Monjardet 2003). Thus, we can introduce a closure operator related to the set of clonal sets of a given relation, and call it *clonal closure operator*.

Proposition 3.1: *Let R be a relation on a set X. The mapping $\mathrm{cl}_R : \mathcal{P}(X) \to \mathcal{C}_R$ defined by*

$$\mathrm{cl}_R(A) = \bigcap\{B \in \mathcal{C}_R \mid A \subseteq B\}$$

is a closure operator on X.

For any subset A of X, $\mathrm{cl}_R(A)$ is called the *clonal closure* of A. Trivially, the clonal closure operator characterizes whether a set is a clonal set or not, i.e. a set A is a clonal set (of a given relation R) if and only if $\mathrm{cl}_R(A) = A$.

Example 3.2: Consider the relation R_B given in Example 2.2. For instance, the clonal closure of $\{e, h\}$, i.e. the smallest clonal set containing $\{e, h\}$, is given by

$$\mathrm{cl}_{R_B}(\{e, h\}) = \{e, f, h, i\},$$

while the clonal closure of $\{e, g\}$ is given by

$$\mathrm{cl}_{R_B}(\{e, g\}) = \{e, f, g, h, i\}.$$

3.2. The clonal lattice

A (non-empty) poset $(L, \leq)$ is called a lattice if any two elements x and y have a greatest lower bound, denoted by $\inf\{x, y\}$ and called the infimum of x and y, and a smallest upper bound, denoted by $\sup\{x, y\}$ and called the supremum of x and y. Similarly, a lattice $(L, \leq)$ is called complete if every subset A of L has both a greatest lower bound, denoted by $\inf A$ and called the infimum of A, and a smallest upper bound, denoted by $\sup A$ and called the supremum of A. A lattice $(L, \leq)$ is called bounded if it has a smallest and a greatest element, respectively denoted by 0 and 1. A non-empty subset M of a lattice $(L, \leq)$ is called a sublattice of L if, for any $x, y \in M$, it holds that $\inf\{x, y\} \in M$ and $\sup\{x, y\} \in M$. A complemented lattice $(L, \leq)$ is a bounded lattice in which any element x has a complement, i.e. there exists an element $y \in L$ such that $\inf\{x, y\} = 0$ and $\sup\{x, y\} = 1$. The relevant notions of distributivity and modularity of a lattice are characterized by means of the lattices N_5 and M_3 illustrated in Figure 5. In particular, a lattice is modular if and only if it has no sublattice of the form N_5, and distributive if and only if it has no sublattice of the form M_3 or N_5. For more details, we refer to Davey and Priestley (2002).

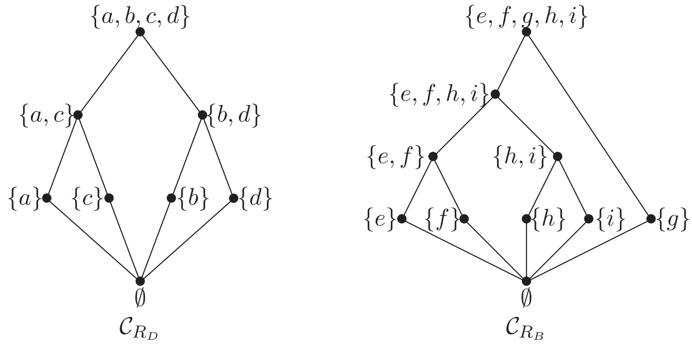

Figure 6. Hasse diagram of the clonal lattice of the relation associated with the diamond graph (left) and of the clonal lattice of the relation associated with the butterfly graph (right).

A closure operator naturally equips the corresponding closure system with a complete lattice structure.

Theorem 3.3 (Davey and Priestley 2002): *Let* cl *be a closure operator on a set X. Then the closure system $\mathcal{E} = \{A \in \mathcal{P}(X) \mid \mathrm{cl}(A) = A\}$ ordered by set inclusion is a complete lattice $(\mathcal{E}, \subseteq)$ in which*

$$\inf_{i\in I} A_i = \bigcap_{i\in I} A_i \text{ and } \sup_{i\in I} A_i = \mathrm{cl}\left(\bigcup_{i\in I} A_i\right),$$

for any family $(A_i)_{i\in I}$ in $\mathcal{E}$, and $0 = \emptyset$ and $1 = X$.

In particular, we conclude that, for any relation R on X, the poset $(\mathcal{C}_R, \subseteq)$ is a complete lattice – called the *clonal lattice* (of R) – in which the infimum is given by the intersection ($\cap$), the supremum is given by the clonal closure of the union ($\mathrm{cl}_R \circ \cup$), and $0 = \emptyset$ and $1 = X$. It is important to mention that, although the clonal lattice is complete, it is not a complete sublattice of $(\mathcal{P}(X), \subseteq)$. In general, the clonal lattice is neither modular, nor complemented, as can be seen in the following examples.

Example 3.4: Consider again the relations R_D and R_B given in Example 2.2. The Hasse diagrams of the clonal lattices of R_D and R_B are shown in Figure 6. Since $N_5 = \{\emptyset, \{b\}, \{c\}, \{b, d\}, X\}$ is a sublattice of $\mathcal{C}_{R_D}$ and $N_5 = \{\emptyset, \{e\}, \{h\}, \{h, i\}, \{e, f, h, i\}\}$ is a sublattice of $\mathcal{C}_{R_B}$, it follows that $\mathcal{C}_{R_D}$ and $\mathcal{C}_{R_B}$ are not modular, and, hence, also not distributive. Note that both $\mathcal{C}_{R_D}$ and $\mathcal{C}_{R_B}$ are complemented (although the complement is not unique).

Example 3.5: Consider the set $X = \{1, 2, 3\}$ equipped with the usual order relation $\leq$. It holds that $\mathcal{C}_\leq = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, X\}$. The Hasse diagram of the clonal lattice of $\leq$ is shown in Figure 7. Since N_5 is a sublattice of $\mathcal{C}_\leq$ (consider, for example, $N_5 = \{\emptyset, \{1\}, \{3\}, \{2, 3\}, X\}$), it follows that $\mathcal{C}_\leq$ is not modular, and, hence, also not distributive. Moreover, it is not complemented either. Indeed, there does not exist a clonal set $A \in \mathcal{C}_\leq$ such that $\{2\} \cap A = \emptyset$ and $\mathrm{cl}_\leq(\{2\} \cup A) = X$.

3.3. Principal filters of the clonal lattice

Next, we study the (principal) filters of the clonal lattice. Recall that a nonempty subset $\mathcal{F}$ of a poset $(P, \leq)$ is called a filter if the following conditions hold:

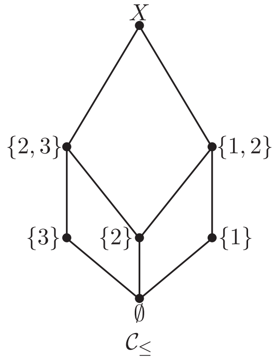

Figure 7. Hasse diagram of the clonal lattice of the order relation $\leq$ in Example 3.5.

(i) for any $x, y \in \mathcal{F}$, there exists an element $z \in \mathcal{F}$ such that $z \leq x$ and $z \leq y$;
(ii) for any $x \in \mathcal{F}$ and $y \in P$, $x \leq y$ implies that $y \in \mathcal{F}$, i.e. $\mathcal{F}$ is an upper set.

The principal filter generated by an element $x \in P$ is the smallest filter that contains x and is given by the set $\{y \in P \mid x \leq y\}$.

In the following, we prove that the principal filters of the clonal lattice are complete sublattices of the clonal lattice in which the supremum operation is now given by the standard union.

Theorem 3.6: *Let R be a relation on a set X, B be a non-empty clonal set of R and $\mathcal{F}_B$ be the principal filter of $(\mathcal{C}_R, \subseteq)$ generated by B, i.e. $\mathcal{F}_B = \{C \in \mathcal{C}_R \mid B \subseteq C\}$. Then it holds that $(\mathcal{F}_B, \subseteq)$ is a complete sublattice of $(\mathcal{C}_R, \subseteq)$ in which*

$$\inf_{i \in I} A_i = \bigcap_{i \in I} A_i \text{ and } \sup_{i \in I} A_i = \bigcup_{i \in I} A_i\,,$$

for any family $(A_i)_{i \in I}$ in $\mathcal{F}_B$, and $0 = B$ and $1 = X$.

***Proof*:** Let $(A_i)_{i \in I}$ be a family in $\mathcal{F}_B$. Since $\mathcal{C}_R$ is a closure system, it follows that $\cap_{i \in I} A_i$ is a clonal set of R. Since $B \subseteq \cap_{i \in I} A_i$, it follows that $\cap_{i \in I} A_i \in \mathcal{F}_B$. Also, from Proposition 2.9, it follows that $\cup_{i \in I} A_i$ is a clonal set of R. Hence, $\left(\text{cl}_R \circ \cup_{i \in I}\right) A_i := \text{cl}_R \left(\cup_{i \in I} A_i\right) = \cup_{i \in I} A_i$. Since $B \subseteq \cup_{i \in I} A_i$, it follows that $\cup_{i \in I} A_i \in \mathcal{F}_B$. Finally, it is clear that $0 = B$ and $1 = X$. □

The above theorem implies that the principal filters of the clonal lattice $(\mathcal{C}_R, \subseteq)$ are complete sublattices of $(\mathcal{P}(X), \subseteq)$.

4. The clonal distance metric

For any integer m, we can define a natural relation expressing which pairs of elements belong to a clonal set of size at most m.

Definition 4.1: Let R be a relation on a set X. For any $m \in \mathbb{N}^*$, the relation φ_R^m on X is defined as

$$\varphi_R^m = \{(x, y) \in X^2 \mid (\exists A \in \mathcal{C}_R)(x, y \in A \wedge |A| \leq m)\}\,.$$

It is straightforward to prove that the relations $(\varphi_R^m)_{m \in \mathbb{N}^*}$ constitute a nested family.

Proposition 4.2: *Let R be a relation on a set X. For any $m \in \mathbb{N}^*$, it holds that $\varphi_R^m \subseteq \varphi_R^{m+1}$.*

Some basic properties of the relation φ_R^m depend on the integer m.

Proposition 4.3: *Let R be a relation on a set X.*

(i) *For any $x, y \in X$, it holds that $x = y$ if and only if $x\varphi_R^1 y$.*
(ii) *For any $x, y \in X$, it holds that $x \approx_R y$ if and only if $x\varphi_R^2 y$.*

***Proof*:**

(i) On the one hand, for any $x, y \in X$ such that $x = y$, it holds that $\{x\} = A \in \mathcal{C}_R$, $x, y \in A$ and $|A| = 1 \leq 1$. Therefore, $x\varphi_R^1 y$. On the other hand, for any $x, y \in X$ such that $x\varphi_R^1 y$, it holds that there exists $A \in \mathcal{C}_R$ satisfying that $x, y \in A$ and $|A| = 1$. Therefore, $x = y$.
(ii) On the one hand, for any $x, y \in X$ such that $x \approx_R y$, we distinguish two cases: $x = y$ and $x \neq y$. In case $x = y$, from (i) we know that $x\varphi_R^1 y$, and, from Proposition 4.2, we conclude that $x\varphi_R^2 y$. In case $x \neq y$, it holds that $\{x, y\} = A \in \mathcal{C}_R$ (and, additionally, $x, y \in A$ and $|A| = 2 \leq 2$). Therefore, $x\varphi_R^2 y$. On the other hand, for any $x, y \in X$ such that $x\varphi_R^2 y$, it holds that there exists $A \in \mathcal{C}_R$ satisfying that $x, y \in A$ and $|A| = 2$. Therefore, $x \approx_R y$. □

Obviously, the relations $(\varphi_R^m)_{m\in\mathbb{N}^*}$ are tolerance relations.

Proposition 4.4: *Let R be a relation on a set X. For any $m \in \mathbb{N}^*$, φ_R^m is a tolerance relation.*

***Proof*:** For any $x \in X$, due to the fact that $\{x\} \in \mathcal{C}_{\mathcal{R}}$ and $|\{x\}| \leq m$, for any $m \geq 1$, it follows that $x\varphi_R^m x$. Hence, φ_R^m is reflexive, for any $m \geq 1$. The symmetry property is evident. We conclude that, for any $m \geq 1$, φ_R^m is a tolerance relation. □

Obviously, as φ_R^1 is the identity relation, it trivially is an equivalence relation, i.e. a relation that is reflexive, symmetric and transitive. However, for any $m \geq 2$, the relation φ_R^m does not necessarily need to be an equivalence relation. For instance, consider an integer n and the set $\{1, \ldots, n\}$ equipped with the usual order relation $\leq$. It holds that $1\varphi_{\leq}^m m$ and $m\varphi_{\leq}^m (2m-1)$, for any $2 \leq m \leq \frac{n+1}{2}$. However, as it does not hold that $1\varphi_{\leq}^m (2m-1)$, we conclude that the transitivity property is not fulfilled.

For defining the clonal distance metric, we assume that the set X is finite in the remainder of this section.

Proposition 4.5: *Let R be a relation on a finite set X of cardinality n. For any $x, y \in X$, it holds that $x\varphi_R^n y$.*

***Proof*:** We recall that $X \in \mathcal{C}_R$. Therefore, for any $x, y \in X$, it holds that $x, y \in A = X$ and $|A| \leq n$. Therefore, $x\varphi_R^n y$. □

Note that the relations $(\varphi_R^m)_{m=1}^n$ can be characterized in terms of the clonal closure of all possible subsets of cardinality two.

Proposition 4.6: *Let R be a relation on a finite set X of cardinality n. For any $x, y \in X$ and any $m \in \mathbb{N}^*$, it holds that $x\varphi_R^m y$ if and only if $|\mathrm{cl}_R(\{x, y\})| \leq m$.*

***Proof*:** Consider $m \in \mathbb{N}^*$ and $x, y \in X$ such that $x\varphi_R^m y$. Hence, there exists $A \in \mathcal{C}_R$ such that $x, y \in A$ and $|A| \leq m$. By definition of the clonal closure of $\{x, y\}$, it is the smallest clonal set containing $\{x, y\}$. Therefore, $\mathrm{cl}_R(\{x, y\}) \subseteq A$ and $|\mathrm{cl}_R(\{x, y\})| \leq m$.

Conversely, consider $m \in \mathbb{N}^*$ and $x, y \in X$ such that $|\text{cl}_R(\{x, y\})| \leq m$. Note that for $A = \text{cl}_R(\{x, y\})$ it holds that $A \in \mathcal{C}_R$, $x, y \in A$ and $|A| \leq m$. Therefore, $x\varphi_R^m y$. □

The corresponding clonal distance metric is then introduced as a tool allowing to compare how far two elements are from being clones.

Definition 4.7: Let R be a relation on a finite set X of cardinality n. For any $x, y \in X$, the clonal distance $d_R(x, y)$ between x and y is defined as

$$d_R(x, y) = \min\{m \in \{1, \ldots, n\} \mid x\varphi_R^m y\} - 1\,.$$

Remark 2: As a consequence of Proposition 4.6, it holds that $d_R(x, y) = |\text{cl}_R(\{x, y\})| - 1$, for any $x, y \in X$.

An important observation concerns the fact that the clonal distance metric effectively constitutes a distance metric on X, thereby justifying its name.

Proposition 4.8: *Let R be a relation on a finite set X of cardinality n. The clonal distance metric $d_R : X \times X \to \mathbb{R}$ defines a distance metric on X.*

Proof: Non-negativity. For any $x, y \in X$, it holds that $x\varphi_R^n y$. Therefore, it holds that $\min\{m \in \{1, \ldots, n\} \mid x\varphi_R^m y\} \geq 1$ and, therefore, $d_R(x, y) \geq 0$.

Identity of indiscernibles. For any $x, y \in X$, it holds that

$$\begin{aligned} d_R(x, y) = 0 &\Leftrightarrow \min\{m \in \{1, \ldots, n\} \mid x\varphi_R^m y\} = 1 \\ &\Leftrightarrow x\varphi_R^1 y \\ &\Leftrightarrow x = y\,. \end{aligned}$$

Symmetry. For any $x, y \in X$, it holds that

$$\begin{aligned} d_R(x, y) &= \min\{m \in \{1, \ldots, n\} \mid x\varphi_R^m y\} - 1 \\ &= \min\{m \in \{1, \ldots, n\} \mid y\varphi_R^m x\} - 1 \\ &= d_R(y, x)\,. \end{aligned}$$

Triangle inequality. For any $x, y, z \in X$, it holds that

$$\text{cl}_R(\{x, z\}) \subseteq \text{cl}_R(\{x, y\} \cup \{y, z\}) \subseteq \text{cl}_R(\{x, y\}) \cup \text{cl}_R(\{y, z\})\,.$$

Removing $\{x\}$ on both sides, it follows that

$$\text{cl}_R(\{x, z\}) \setminus \{x\} \subseteq (\text{cl}_R(\{x, y\}) \setminus \{x\}) \cup (\text{cl}_R(\{y, z\}) \setminus \{x\})\,.$$

We conclude that

$$\begin{aligned} d_R(x,z) &= |\mathrm{cl}_R(\{x,z\})| - 1 \\ &= |\mathrm{cl}_R(\{x,z\})\setminus\{x\}| \\ &\leq |(\mathrm{cl}_R(\{x,y\})\setminus\{x\}) \cup (\mathrm{cl}_R(\{y,z\})\setminus\{x\})| \\ &\leq |\mathrm{cl}_R(\{x,y\})\setminus\{x\}| + |\mathrm{cl}_R(\{y,z\})\setminus\{x\}| - |\{y\}| \\ &\leq d_R(x,y) + d_R(y,z)\,. \end{aligned}$$

□

Example 4.9: Consider again the relations R_D and R_B given in Example 2.2. The clonal distance metrics associated with both relations are represented in the following tables:

d_{R_D}	a	b	c	d
a	0	3	1	3
b	3	0	3	1
c	1	3	0	3
d	3	1	3	0

d_{R_B}	e	f	g	h	i
e	0	1	4	3	3
f	1	0	4	3	3
g	4	4	0	4	4
h	3	3	4	0	1
i	3	3	4	1	0

5. Some special cases

In case the given relation is an equivalence relation or a weak/total order relation, its clonal sets can be easily characterized. In the former case, a clonal set is either a subset of a unique equivalence class or the union of two or more equivalence classes. Similarly, in the latter case, a clonal set is either a subset of a unique equivalence class or the union of two or more consecutive equivalence classes.

5.1. Equivalence relations

For a given equivalence relation E on a set X, the equivalence class of an element $x \in X$ is defined by $[x]_E = \{y \in X \mid xEy\}$.

Proposition 5.1: *Let E be an equivalence relation on a set X. A subset A of X is a clonal set of E if and only if it is either a subset of an equivalence class of E or the union of two or more equivalence classes of E.*

Proof: $\Rightarrow$ Let $A \in \mathcal{C}_E$. If $A = \emptyset$, then it clearly holds that $A \subset [x]_E$, for any $x \in X$. If $A \neq \emptyset$, then there exists a such that $a \in A$. We distinguish two cases:

(i) For any $x \in A$, it holds that xEa. It immediately follows that $A \subseteq [a]_E$.
(ii) There exists $b \in A$ such that $bE^c a$.

(a) We prove that $[a]_E \subseteq A$ and $[b]_E \subseteq A$. Assume that $[a]_E \not\subseteq A$ or $[b]_E \not\subseteq A$. If $[a]_E \not\subseteq A$, then it follows that there exists c such that $cEa \wedge c \notin A$. Since $A \in \mathcal{C}_E$, $c \in X \setminus A$, $a, b \in A$ and cEa, it holds that cEb. Since E is an equivalence relation, it follows that bEa, a contradiction. The other case is proved analogously. We conclude that $[a]_E \subseteq A$, $[b]_E \subseteq A$ and $[a]_E \neq [b]_E$.

(b) We prove that, for any $x \in A$, it holds that $[x]_E \subseteq A$. The result is already proved for $x = a$. Consider $x \neq a$ and suppose that there exists $x_0 \in X$ such that $x_0 \in [x]_E$ and $x_0 \notin A$. Since $A \in \mathcal{C}_E$, $x_0 \in (X \setminus A)$, $x, a \in A$ and x_0Ex, it follows that x_0Ea. Hence, $x_0 \in [a]_E$. Since $[a]_E \subseteq A$, it follows that $x_0 \in A$, a contradiction. We conclude that, for any $x \in A$, $[x]_E \subseteq A$. Hence, $\cup_{x\in A}[x]_E \subseteq A$, and, obviously, $A = \cup_{x\in A}[x]_E$.

Since $a, b \in A$, $[a]_E \neq [b]_E$ and $A = \cup_{x\in A}[x]_E$, we conclude that A is the union of two or more equivalence classes of E.

$\Leftarrow$ Let $A \subseteq X$.

(a) If A is a subset of an equivalence class $[a]_E$, then for any $x, y \in A$ and any $z \in X \setminus A$, it follows that $x, y \in [a]_E$, which implies that $zEx \Leftrightarrow zEy$ and $xEz \Leftrightarrow yEz$. Hence, $A \in \mathcal{C}_E$.

(b) If A is the union of two or more equivalence classes, then for any $x, y \in A$ and any $z \in X \setminus A$, there exist a, b such that $x \in [a]_E \subseteq A$ and $y \in [b]_E \subseteq A$ (note that $[a]_E$ might be equal to $[b]_E$). This implies that zE^cx, zE^cy, xE^cz and yE^cz (otherwise $z \in [a]_E \cup [b]_E$, and hence $z \in A$, a contradiction). Thus, $zEx \Leftrightarrow zEy$ and $xEz \Leftrightarrow yEz$. Hence, $A \in \mathcal{C}_E$.

□

Example 5.2: Consider the equivalence relation E on the set $X = \{a, b, c, d, e, f\}$ consisting of three equivalence classes: $[a]_E$, $[b]_E = [c]_E$ and $[d]_E = [e]_E = [f]_E$. The set of clonal sets of E is given by:

$$\mathcal{C}_E = \left\{ \begin{array}{l} \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{b, c\}, \{d, e\}, \{d, f\}, \{e, f\}, \\ \{a, b, c\}, \{d, e, f\}, \{a, d, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\} \end{array} \right\},$$

resulting in the clonal lattice displayed in Figure 8.

The clonal distance metric is represented in the following table:

d_E	a	b	c	d	e	f
a	0	2	2	3	3	3
b	2	0	1	4	4	4
c	2	1	0	4	4	4
d	3	4	4	0	1	1
e	3	4	4	1	0	1
f	3	4	4	1	1	0

5.2. Weak order relations

A relation $\lesssim$ on a set X is called a weak order relation if it is reflexive, transitive and complete. Note that a weak order relation can be understood as a total order relation in

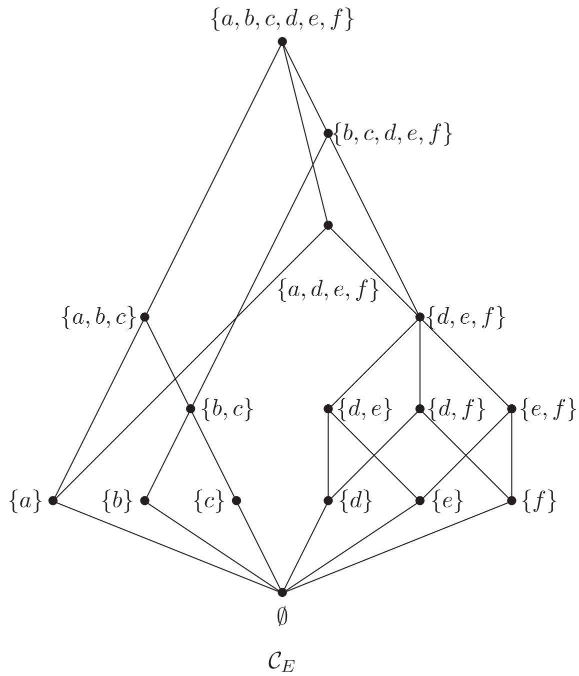

Figure 8. Hasse diagram of the clonal lattice of the equivalence relation E in Example 5.2.

which ties between consecutive elements are allowed. Any weak order relation $\lesssim$ can be partitioned in two relations

$$< = \{(x,y) \in X^2 \mid x \lesssim y \wedge y \lesssim^c x\},$$
$$\sim = \{(x,y) \in X^2 \mid x \lesssim y \wedge y \lesssim x\},$$

where the relation $<$ is irreflexive, transitive and antisymmetric and the relation $\sim$ is an equivalence relation.

An equivalence class $[a]_\sim$ is said to be strictly smaller than another equivalence class $[b]_\sim$ if, for any $x \in [a]_\sim$ and any $y \in [b]_\sim$, it holds that $x < y$. Analogously, an equivalence class $[a]_\sim$ is said to be strictly greater than another equivalence class $[b]_\sim$ if, for any $x \in [a]_\sim$ and any $y \in [b]_\sim$, it holds that $y < x$. Two equivalence classes $[a]_\sim$ and $[b]_\sim$ are said to be consecutive if there does not exist an equivalence class $[c]_\sim$ such that, for any $x \in [a]_\sim$, any $y \in [b]_\sim$ and any $z \in [c]_\sim$, it holds that $x < z < y$ or $y < z < x$.

Proposition 5.3: *Let $\lesssim$ be a weak order relation on a set X. A subset A of X is a clonal set of $\lesssim$ if and only if it is either a subset of an equivalence class of $\sim$ or the union of two or more consecutive equivalence classes of $\sim$.*

Proof: $\Rightarrow$ Let $A \in \mathcal{C}_{\lesssim}$. If $A = \emptyset$, then it clearly holds that $A \subset [x]_\sim$, for any $x \in X$. If $A \neq \emptyset$, then there exists a such that $a \in A$. We distinguish two cases:

(i) For any $x \in A$, it holds that $x \sim a$. It immediately follows that $A \subseteq [a]_\sim$.
(ii) There exists $b \in A$ such that $b \sim^c a$. We will prove that A is the union of two or more consecutive equivalence classes.

(a) We prove that $[a]_\sim \subseteq A$ and $[b]_\sim \subseteq A$ (obviously, $[a]_\sim \neq [b]_\sim$). Suppose that there exists c such that $c \sim a$ (thus $c \lesssim a$ and $a \lesssim c$) and $c \notin A$. Since $A \in \mathcal{C}_{\lesssim}$, $c \in X \backslash A$, $a, b \in A$ and $c \lesssim a$ and $a \lesssim c$, it follows that $c \lesssim b$ and $b \lesssim c$. This implies that $b \sim c$ and, therefore, $a \sim b$, a contradiction. We conclude that $[a]_\sim \subseteq A$. The proof for $[b]_\sim \subseteq A$ is analogous.

(b) We prove that, for any $x \in A$, it holds that $[x]_\sim \subseteq A$. Let $x \in A$ and suppose that there exists $x_0 \in [x]_\sim$ such that $x_0 \notin A$. The result is already proved for $x = a$, therefore, we prove it for $x \neq a$. Since $A \in \mathcal{C}_{\lesssim}$, $x_0 \in X \backslash A$, $x, a \in A$ and $x_0 \lesssim x$ and $x \lesssim x_0$, it follows that $x_0 \lesssim a$ and $a \lesssim x_0$. This implies that $a \sim x_0$ and, therefore, $x_0 \in [a]_\sim \subseteq A$, a contradiction.

We conclude that A is the union of two or more equivalence classes of $\lesssim$. We now prove that all equivalence classes are consecutive. Suppose that there exists $a, b \in A$ and $c \notin A$ such that $a \lesssim c \lesssim b$, $b \lesssim^c a$, $b \lesssim^c c$ and $c \lesssim^c a$. Since $A \in \mathcal{C}_{\lesssim}$, $a, b \in A$, $c \notin A$ and $a \lesssim c$, it follows that $b \lesssim c$, a contradiction.

$\Leftarrow$ Let $A \subseteq X$.

(a) If A is a subset of an equivalence class $[a]_\sim$, then for any $x, y \in A$ and any $z \in X \setminus A$, it immediately follows that $z \lesssim x \Leftrightarrow z \lesssim y$ and $x \lesssim z \Leftrightarrow y \lesssim z$. Hence, $A \in \mathcal{C}_{\lesssim}$.

(b) If A is the union of two or more consecutive equivalence classes, then, for any $x, y \in A$ and any $z \in X \setminus A$, z belongs to an equivalence class strictly greater than the equivalence classes of x and y or to an equivalence class strictly smaller than the equivalence classes of x and y. Thus, $z \lesssim x \Leftrightarrow z \lesssim y$ and $x \lesssim z \Leftrightarrow y \lesssim z$. Hence, $A \in \mathcal{C}_{\lesssim}$.

□

Corollary 5.4: *Let $\leq$ be a total order relation on a set X. A non-empty subset A of X is a clonal set of $\leq$ if and only if it is a set of consecutive elements of $\leq$.*

Note that the definition of a clonal set coincides with that commonly accepted in social choice theory (Tideman 1987; Zavist and Tideman 1989) when restricted to total order relations or rankings (strict total order relations).[3] However, in case weak order relations or rankings with ties are considered, both definitions slightly differ since in social choice theory a clonal set is characterized as an equivalence class of $\sim$ or the union of two or more consecutive equivalence classes of $\sim$ (note that proper subsets of an equivalence class of $\sim$ are not considered to be clonal sets).

Example 5.5: Consider the weak order relation $\lesssim$ on the set $X = \{a, b, c, d, e, f\}$ in which $a < b \sim c < d \sim e \sim f$. The set of clonal sets of $\lesssim$ is given by:

$$\mathcal{C}_{\lesssim} = \left\{ \begin{array}{l} \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{b, c\}, \{d, e\}, \{d, f\}, \{e, f\}, \\ \{a, b, c\}, \{d, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\} \end{array} \right\},$$

resulting in the clonal lattice displayed in Figure 9. It must be remarked that only $\{b, c\}$, $\{a, b, c\}$, $\{d, e, f\}$, $\{b, c, d, e, f\}$ and $\{a, b, c, d, e, f\}$ are clonal sets in the sense of social choice theory.

The corresponding clonal distance metric is represented in the following table:

$d_{\lesssim}$	a	b	c	d	e	f
a	0	2	2	5	5	5
b	2	0	1	4	4	4
c	2	1	0	4	4	4
d	5	4	4	0	1	1
e	5	4	4	1	0	1
f	5	4	4	1	1	0

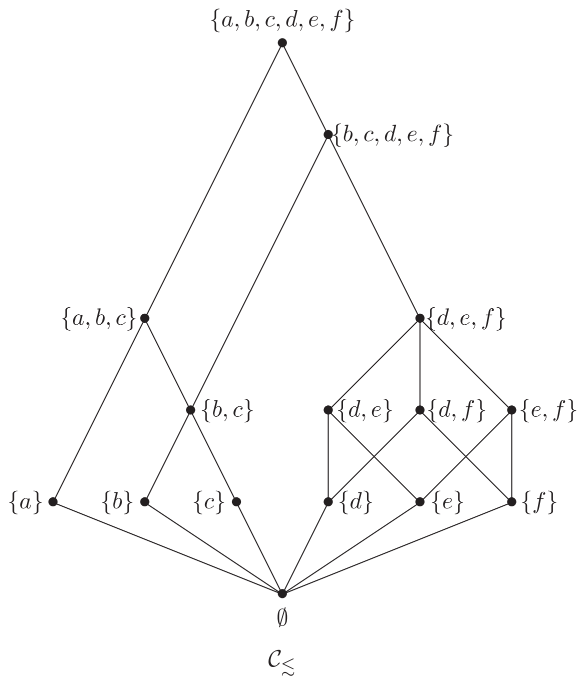

Figure 9. Hasse diagram of the clonal lattice of the weak order relation $\lesssim$ in Example 5.5.

6. Conclusions

In this work, we have generalized the notion of clone relation of a binary relation into the notion of clonal set. In particular, pairs of elements related by the clone relation turn out to be clonal sets of cardinality two. This generalization leads to many interesting results, such as the fact that the set of clonal sets is a complete lattice with the intersection and the clonal closure of the union as infimum and supremum. Also, the introduction of the clonal distance metric results in a natural way of turning a set equipped with a relation into a metric space.

Notes

1. Note that the term clone is also an old acquaintance of algebraists (Post 1941), however, carrying a totally different meaning.
2. In this paper, a relation R is identified with its characteristic mapping χ_R, i.e. $\chi_R(x, y) = 1$ means that xRy and $\chi_R(x, y) = 0$ means that xR^cy. In a finite setting, a relation can be conveniently represented as a matrix such that $R_{ij} = \chi_R(x_i, x_j)$.

3. Actually, in social choice theory it is required that the cardinality of the set is greater than or equal to two. Thus, the empty set and all singletons are not clonal sets in the sense of social choice theory.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

Raúl Pérez-Fernández is supported as a postdoc by the Research Foundation of Flanders (Fonds Wetenschappelijk Onderzoek) [FWO17/PDO/160].

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