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Left- and right-compatibility of order relations and fuzzy tolerance relations

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Abstract

In a recent paper, De Baets et al. have studied the compatibility of a(n) (strict) order relation with a fuzzy relation, and have characterized the fuzzy tolerance (and, in particular, fuzzy equivalence) relations that a given strict order relation is compatible with. We extend this study by considering the left- and right-compatibility of a(n) (strict) order relation with a fuzzy tolerance relation and vice versa. We characterize the fuzzy tolerance relations that are compatible with a given (strict) order relation. Conversely, we provide a representation of the fuzzy tolerance relations that a given strict order relation is left- or right-compatible with. Specific attention is paid to the case of fuzzy equivalence relations. We conclude by pointing out that the representation theorems in the above-mentioned paper need some minor rectification.

Keywords: Left-compatibility; Right-compatibility; Clone relation; Order relation; Fuzzy equivalence relation; Fuzzy tolerance relation

1. Introduction

Bělohlávek [1] introduced the notion of compatibility of a fuzzy relation R with a fuzzy equivalence relation E on a universe of discourse X and used it in his definition of a fuzzy order. Compatibility expresses that elements that are similar to related elements are related as well, i.e.,

$$R(x_1, y_1) * E(x_1, x_2) * E(y_1, y_2) \leq R(x_2, y_2)\,, \tag{1}$$

for any x_1, x_2, y_1, y_2 in X. Note that this notion of compatibility is not symmetric in general. Kheniche et al. [17] have pointed out that it is equivalent to the older notion of extensionality of a mapping between two universes endowed with fuzzy equalities introduced by Höhle and Blanchard [16]. The importance of this notion stems from its abundant use, e.g. in the study of fuzzy functions [8,13,20], fuzzy ordered structures [1,2,14,19,22,24] and fuzzy algebras [3,13,15].

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Some of the present authors have shown recently [11] that the above notion of compatibility needs to be handled with care. Indeed, already in the basic setting when R is a (crisp) order relation, it turns out that it is only compatible with the crisp equality relation. However, it suffices to consider the corresponding strict order relation to realize that there might exist a multitude of fuzzy tolerance relations it is compatible with. Indeed, by introducing the notion of clone relation associated with a(n) (strict) order relation, we do not only have a tool to check for the existence of such fuzzy tolerance relations, but it also gives us a means to provide a representation of all of them. Recently, this notion of clone relation has been further extended to the case of a general binary relation [7].

In the present paper we further dissect the notion of compatibility into left- and right-compatibility and investigate these relations between (strict) order relations on the one hand, and fuzzy tolerance relations on the other hand. Since also these compatibility notions are not symmetric, two subproblems need to be considered: the left- and right-compatibility of a(n) (strict) order relation with a fuzzy tolerance relation, and conversely, the left- and right-compatibility of a fuzzy tolerance relation with a(n) (strict) order relation. More specifically, we provide characterizations of the fuzzy tolerance relations that are left- or right-compatible (which turn out to be equivalent in this case) with a given (strict) order relation. Conversely, we provide a representation of the fuzzy tolerance relations that a given (strict) order relation is left- or right-compatible with. Where possible, we further refine the results for the particular case of fuzzy equivalence relations, i.e. transitive fuzzy tolerance relations. Along the way, we point out that the representation theorems (Theorems 2–4 and some of their corollaries) in [11] are not fully correct, and provide minor rectifications thereof.

This paper is organized as follows. After recalling some basic definitions and properties of ordered sets and fuzzy relations in Section 2, we study the left- and right-compatibility of fuzzy tolerance relations with a given order relation in Section 3. In Section 4, we show that left- and right-compatibility of an order relation with a fuzzy tolerance relation are void concepts, subsequently turning our attention to the more interesting case of strict order relations in Section 5. In that section, we generalize the notion of clone relation of an order relation to the notions of left-clone and right-clone relations, and we extend the main representation theorem of fuzzy tolerance relations that a given strict order relation is compatible with to the case of left- and right-compatibility. In each section, we pay particular attention to the case of fuzzy equivalence relations. Finally, we present some concluding remarks in Section 6.

2. Preliminaries

This section serves an introductory purpose. First, we recall some basic concepts related to partially ordered sets and lattices, and to fuzzy relations. Second, we recall some basic definitions and results on the left- and right-compatibility of two fuzzy relations. Further information can be found in [4,9,17,21].

2.1. Basic definitions

A binary crisp relation (or a relation, for short) R on a set X is a subset of X^2. We usually write xRy instead of $(x, y) \in R$. We denote by R^t the *transpose* (*converse*) of R, i.e., for any $x, y \in X$, $xR^t y$ means that yRx. Also, by R^c we denote the *complement* of R, i.e., for any $x, y \in X$, $xR^c y$ means that $(x, y) \notin R$. The composition of two relations R and S on X is the relation $R \circ S$ on X defined as

$$R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}.$$

A set X equipped with an order relation $\leqslant$ is called a partially ordered set (poset, for short), denoted $(X, \leqslant)$. Two elements x and y of a poset $(X, \leqslant)$ are called incomparable, denoted $x \parallel y$, if $\neg(x \leqslant y)$ and $\neg(y \leqslant x)$; otherwise, they are called comparable, denoted $x \nparallel y$. Further, $\{x, y\}^u$ denotes the set of all upper bounds of x and y, while $\{x, y\}^l$ denotes the set of all lower bounds of x and y, i.e., $\{x, y\}^u = \{z \in X \mid x \leqslant z \ \wedge \ y \leqslant z\}$ and $\{x, y\}^l = \{z \in X \mid z \leqslant x \ \wedge \ z \leqslant y\}$.

A strict order relation $<$ on a set X is a relation that is irreflexive (i.e., $x < x$ does not hold for any $x \in X$) and transitive, implying that it is asymmetric (i.e., $x < y$ implies $\neg(y < x)$, for any $x, y \in X$). To any order relation $\leqslant$ corresponds a strict order relation $<$ (its strict part or irreflexive kernel): $x < y$ if $x \leqslant y$ and $x \neq y$. Conversely, to any strict order relation $<$ corresponds an order relation $\leqslant$ (its reflexive closure): $x \leqslant y$ if $x < y$ or $x = y$. The covering relation $\ll$ of a poset $(X, \leqslant)$ is the binary relation on X defined by: $x \ll y$ if $x < y$ and there exists no $z \in X$ such that $x < z < y$.

A poset $(L, \leq)$ is called a $\wedge$-semi-lattice if any two elements x and y have a greatest lower bound, denoted $x \wedge y$ and called the meet (infimum) of x and y; it is called a $\vee$-semi-lattice if any two elements x and y have a smallest upper bound, denoted $x \vee y$ and called the join (supremum) of x and y. A poset $(L, \leq)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice. A poset $(L, \leq)$ is called bounded if it has a smallest and a greatest element, respectively denoted by 0 and 1. Throughout this paper, unless otherwise stated, L always denotes a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$ and $*$ a t-norm on it, i.e. an increasing, commutative and associative binary operation with neutral element 1. T-norms were originally introduced on the real unit interval $[0, 1]$, but are readily extended to posets and lattices [12]. For more details on t-norms on bounded lattices, we refer to [10,18].

A binary fuzzy relation or a binary L-relation (L-relation, for short) R on a universe X is a mapping $R : X \times X \to L$. If $L = \{0, 1\}$, crisp relations are obtained. The following properties of L-relations are of interest in this paper:

(i) reflexivity: $R(x, x) = 1$, for any $x \in X$;
(ii) symmetry: $R(x, y) = R(y, x)$, for any $x, y \in X$;
(iii) $*$-transitivity: $R(x, y) * R(y, z) \leq R(x, z)$, for any $x, y, z \in X$;
(iv) separability: $R(x, y) = 1$ implies $x = y$, for any $x, y \in X$.

For further details, we refer to [1,5,6,23].

An L-relation E on X is called an L-tolerance relation if it is reflexive and symmetric; a $*$-transitive L-tolerance relation is called an L-equivalence relation and a separable L-equivalence relation is called an L-equality relation. Note that the crisp equality $\delta = \{(x, y) \in X^2 \mid x = y\}$ is the only $\{0, 1\}$-equality relation on X and it is the smallest equivalence relation on X.

An L-relation R_1 is said to be included in an L-relation R_2, denoted $R_1 \subseteq R_2$, if $R_1(x, y) \leq R_2(x, y)$, for any $x, y \in X$. The intersection of two L-relations R_1 and R_2 on X is the L-relation $R_1 \cap R_2$ on X defined by $R_1 \cap R_2(x, y) = R_1(x, y) \wedge R_2(x, y)$, for any $x, y \in X$. Similarly, the union of two L-relations R_1 and R_2 on X is the L-relation $R_1 \cup R_2$ on X defined by $R_1 \cup R_2(x, y) = R_1(x, y) \vee R_2(x, y)$, for any $x, y \in X$. The transpose R^t of an L-relation R is defined by $R^t(x, y) = R(y, x)$.

2.2. *Left-compatibility, right-compatibility and compatibility of L-relations*

The notion of compatibility of an L-relation on X with respect to (with, for short) an L-equality relation on X has been introduced by Bělohlávek [2]. In [17], we showed that this notion is equivalent to the older notion of extensionality introduced by Höhle and Blanchard [16]. Also, we generalized this notion to arbitrary L-relations.

Definition 1. [17] Let R_1 and R_2 be two L-relations on X.

(i) R_1 is called left-compatible with R_2 if it holds that

$$R_1(x, y) * R_2(x, z) \leq R_1(z, y) \ , \tag{2}$$

for any $x, y, z \in X$;

(ii) R_1 is called right-compatible with R_2 if it holds that

$$R_1(x, y) * R_2(y, t) \leq R_1(x, t) \ , \tag{3}$$

for any $x, y, t \in X$;

(iii) R_1 is called compatible with R_2 if it holds that

$$R_1(x, y) * R_2(x, z) * R_2(y, t) \leq R_1(z, t) \ , \tag{4}$$

for any $x, y, z, t \in X$.

Note that the relation $R = X^2$ is left-compatible, right-compatible and compatible with any L-relation on X.

In what follows, we will use the following lemma and proposition.

Lemma 1. [17] *For any two L-relations R_1 and R_2 on X, the following equivalences hold:*

(i) *R_1 is left-compatible with R_2 if and only if R_1^t is right-compatible with R_2;*
(ii) *R_1 is right-compatible with R_2 if and only if R_1^t is left-compatible with R_2;*
(iii) *R_1 is compatible with R_2 if and only if R_1^t is compatible with R_2.*

Proposition 1. *Let R_1 and R_2 be two L-relations on X. Then it holds that*

(i) *If R_1 is left- and right-compatible with R_2, then R_1 is compatible with R_2;*
(ii) *If R_1 is compatible with R_2 and R_2 is reflexive, then R_1 is left- and right-compatible with R_2.*

Note that throughout this paper, we use the notation τ to refer to the characteristic mapping of a relation. In particular,

$$\tau(x \leqslant y) = \begin{cases} 1 & \text{, if } x \leqslant y\,; \\ 0 & \text{, if } x \nleqslant y\,. \end{cases}$$

Also, to avoid any confusion, we emphasize that the symbol $\leqslant$ refers to a partial order relation on X, while the symbol $\leq$ refers to the partial order relation of the lattice L.

3. Left- and right-compatibility of an L-tolerance or L-equivalence relation with an order relation

3.1. Basic results

The following proposition shows that all types of compatibility of an L-tolerance relation with an order relation are equivalent.

Proposition 2. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. Then the following statements are equivalent:*

(i) *E is left-compatible with $\leqslant$;*
(ii) *E is right-compatible with $\leqslant$;*
(iii) *E is compatible with $\leqslant$.*

Proof. (i) ⇒ (ii): Since E is left-compatible with $\leqslant$, it follows from Lemma 1(i) that E^t is right-compatible with $\leqslant$. As E is symmetric, it holds that E is right-compatible with $\leqslant$.

(ii) ⇒ (iii): Since E is symmetric and E is right-compatible with $\leqslant$, it follows that

$$\begin{aligned} E(x,y) * \tau(x \leqslant z) * \tau(y \leqslant t) &= E(y,x) * \tau(x \leqslant z) * \tau(y \leqslant t) \\ &\leq E(y,z) * \tau(y \leqslant t) \\ &= E(z,y) * \tau(y \leqslant t) \\ &\leq E(z,t)\,, \end{aligned}$$

for any $x, y, z, t \in X$. Hence, E is compatible with $\leqslant$.

(iii) ⇒ (i): Follows from Proposition 1(ii). □

Since the three types of compatibility of any L-tolerance (and, in particular, L-equivalence) relation with an order relation are equivalent, in the following two subsections, we only refer to compatibility while using the convenient simpler inequality

$$E(x,y) * \tau(x \leqslant z) \leq E(z,y)\,, \tag{5}$$

for any $x, y, z \in X$.

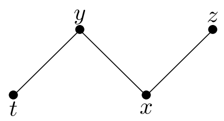

Fig. 1. Hasse diagram of the poset $(X, \leqslant)$ with $X = \{x, y, z, t\}$.

Moreover, the left-compatibility (resp. right-compatibility) of any L-relation with a given strict order relation $<$ is equivalent to the left-compatibility (resp. right-compatibility) with the corresponding order relation $\leqslant$.

Proposition 3. *Let $(X, \leqslant)$ be a poset, $<$ be the corresponding strict order relation and E be an L-relation on X. Then it holds that*

(i) *E is left-compatible with $\leqslant$ if and only if E is left-compatible with $<$;*
(ii) *E is right-compatible with $\leqslant$ if and only if E is right-compatible with $<$;*
(iii) *E is compatible with $\leqslant$ if and only if E is left- and right-compatible with $<$.*

Proof. (i) Suppose that E is left-compatible with $\leqslant$. Let $x, y, z \in X$. Since $\tau(x < z) \leq \tau(x \leqslant z)$, it holds that $E(x, y) * \tau(x < z) \leq E(x, y) * \tau(x \leqslant z)$. Hence, $E(x, y) * \tau(x < z) \leq E(z, y)$. Thus, E is left-compatible with $<$.
Conversely, the fact that E is left-compatible with $<$ and with the crisp equality then implies that

$$E(x, y) * \tau(x \leqslant z) = E(x, y) * (\tau(x < z) \vee \tau(x = z)) \leq E(z, y)\,,$$

for any $x, y, z \in X$. Thus, E is left-compatible with $\leqslant$.
(ii) Follows from Lemma 1 and (i).
(iii) Follows from Proposition 1, (i) and (ii). □

3.2. Compatibility of an L-tolerance relation with an order relation

In this subsection, we provide a characterization of the L-tolerance relations that are compatible with a given order relation. First, we introduce the following binary relations $\triangledown$ and $\vartriangle$ on X associated with the poset $(X, \leqslant)$:

$$\triangledown = \{(x, y) \in X^2 \mid \{x, y\}^l \neq \emptyset\}\,,$$
$$\vartriangle = \{(x, y) \in X^2 \mid \{x, y\}^u \neq \emptyset\}\,.$$

Also, we will use the following notation:

$$\boxtimes = \triangledown \cup \vartriangle = \{(x, y) \in X^2 \mid \{x, y\}^l \neq \emptyset \vee \{x, y\}^u \neq \emptyset\}\,.$$

Obviously, it holds that $(\leqslant \cup \leqslant^t) \subseteq (\triangledown \cap \vartriangle)$. Clearly, $\triangledown$, $\vartriangle$ and $\boxtimes$ are tolerance relations. In general, these relations are not transitive, as is the case for the poset of Example 1.

Example 1. Let $(X, \leqslant)$ be the poset given by the Hasse diagram in Fig. 1.
It holds that

$$\triangledown = \delta \cup \{(x, y), (y, x), (x, z), (z, x), (t, y), (y, t), (y, z), (z, y)\}\,,$$
$$\vartriangle = \delta \cup \{(x, y), (y, x), (x, z), (z, x), (t, y), (y, t), (t, x), (x, t)\}\,.$$

The following proposition is straightforward.

Proposition 4. *Let $(X, \leqslant)$ be a poset and consider the relations $\triangledown$, $\vartriangle$ and $\boxtimes$. Then it holds that*

(i) *If $(X, \leqslant)$ is a $\wedge$-semi-lattice (resp. a $\vee$-semi-lattice), then $\triangledown = X^2$ (resp. $\vartriangle = X^2$).*
(ii) *If $(X, \leqslant)$ has a smallest (resp. a greatest) element, then $\triangledown = X^2$ (resp. $\vartriangle = X^2$).*
(iii) *If $(X, \leqslant)$ is a bounded poset or a lattice, then $\triangledown = \vartriangle = \boxtimes = X^2$.*

The following proposition shows that the tolerance relation $\triangledown$ is compatible with $\leqslant$.

Proposition 5. *The relation $\triangledown$ associated with a poset $(X, \leqslant)$ is compatible with $\leqslant$.*

Proof. Let $x, y, z \in X$, then we need to prove that

$$\tau(x \triangledown y) * \tau(x \leqslant z) \leq \tau(z \triangledown y) .$$

If $x \triangledown^c y$, then this inequality trivially holds. If $x \triangledown y$, then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. Hence,

$$\tau(x \triangledown y) * \tau(x \leqslant z) \leq (\tau(c \leqslant x) \wedge \tau(c \leqslant y)) * \tau(x \leqslant z) .$$

The fact that $* \leq \wedge$ implies that

$$(\tau(c \leqslant x) \wedge \tau(c \leqslant y)) * \tau(x \leqslant z) \leq \tau(c \leqslant z) \wedge \tau(c \leqslant y) \leq \tau(z \triangledown y) .$$

Hence,

$$\tau(x \triangledown y) * \tau(x \leqslant z) \leq \tau(z \triangledown y) .$$

Thus, $\triangledown$ is compatible with $\leqslant$. $\square$

Remark 1. The relations $\vartriangle$ and $\boxtimes$ associated with a poset $(X, \leqslant)$ are not necessarily compatible with $\leqslant$. Indeed, consider the poset of Example 1. It is clear that

$$\underbrace{\tau(x \vartriangle t)}_{=1} * \underbrace{\tau(x \leqslant z)}_{=1} \not\leq \underbrace{\tau(z \vartriangle t)}_{=0} .$$

This implies that $\vartriangle$ is not compatible with $\leqslant$. Similarly, it follows that $\boxtimes$ is not compatible with $\leqslant$.

The following proposition establishes an interesting necessary condition.

Proposition 6. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. If E is compatible with $\leqslant$, then $\triangledown \subseteq E$.*

Proof. Suppose that E is compatible with $\leqslant$ and consider $x, y \in X$ such that $x \triangledown y$. Then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. Due to the compatibility of E with $\leqslant$ and the reflexivity of E, it holds that

$$\underbrace{E(c, c)}_{=1} * \underbrace{\tau(c \leqslant x)}_{=1} \leq E(c, x) .$$

Hence, $E(c, x) = 1$. Again, the symmetry of E and its compatibility with $\leqslant$ imply that

$$\underbrace{E(x, c)}_{=E(c,x)=1} * \underbrace{\tau(c \leqslant y)}_{=1} \leq E(x, y) .$$

Hence, $E(x, y) = 1$, and thus $\triangledown \subseteq E$. $\square$

Proposition 6 states a necessary condition for the compatibility of an L-tolerance relation E with an order relation $\leqslant$. However, this condition is not sufficient, as can be seen in Example 2. In case E is an L-equivalence relation, we will show in the next subsection that this condition becomes necessary and sufficient.

Example 2. Consider the poset of Example 1. Consider the tolerance relation E on X defined as: $E = X^2 \setminus \{(z, t), (t, z)\}$, then it holds that $\triangledown \subseteq E$. However, since

$$\underbrace{\tau(E(x, t))}_{=1} * \underbrace{\tau(x \leqslant z)}_{=1} \not\leq \underbrace{\tau(E(z, t))}_{=0} ,$$

it is clear that E is not compatible with $\leqslant$.

Propositions 5 and 6 imply that the relation $\triangledown$ is the smallest L-tolerance relation on X that is compatible with $\leqslant$, while X^2 is the greatest one.

In view of Proposition 4, we obtain the following corollary. It shows that under mild conditions, the compatibility of an L-tolerance relation with an order relation is a trivial notion.

Corollary 1. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. If $(X, \leqslant)$ is a $\wedge$-semi-lattice or has a smallest element, then it holds that E is compatible with $\leqslant$ if and only if $E = X^2$.*

Definition 2. Let $(X, \leqslant)$ be a poset, Y be a nonempty subset of X^2 and R be an L-relation on X. R is called increasing on Y if for any $(x, y), (z, t) \in Y$ such that $x \leqslant z$ and $y \leqslant t$, it holds that $R(x, y) \leq R(z, t)$.

The following theorem characterizes the L-tolerance relations that are compatible with a given order relation.

Theorem 1. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. Then it holds that E is compatible with $\leqslant$ if and only if there exists an L-tolerance relation α on X with $\alpha \subseteq \triangledown^c \cup \delta$ such that α is increasing on $\triangledown^c$ and $E = \triangledown \cup \alpha$.*

Proof. Suppose that E is compatible with $\leqslant$, then according to Proposition 6 it holds that $\triangledown \subseteq E$. Consider the L-relation α on X defined as $\alpha = (E \setminus \triangledown) \cup \delta = (E \cap \triangledown^c) \cup \delta = E \cap (\triangledown^c \cup \delta)$, i.e.,

$$\alpha(x, y) = \begin{cases} E(x, y) & \text{, if } (x, y) \in \triangledown^c \cup \delta\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

It is obvious that $\alpha \subseteq \triangledown^c \cup \delta$ and $E = \triangledown \cup \alpha$. It remains to show that α is an L-tolerance relation on X that is increasing on $\triangledown^c$. Note that if $(x, y) \in \triangledown^c \cup \delta$, then it also holds that $(y, x) \in \triangledown^c \cup \delta$. Hence, α is an L-tolerance relation on X. Due to the symmetry of α, in order to show that α is increasing on $\triangledown^c$, it suffices to show that $x \leqslant z$ implies that $\alpha(x, y) \leq \alpha(z, y)$, for any $(x, y), (z, y) \in \triangledown^c$. Let $(x, y), (z, y) \in \triangledown^c$ such that $x \leqslant z$, then it holds that $\alpha(x, y) = E(x, y)$ and $\alpha(z, y) = E(z, y)$. The compatibility of E with $\leqslant$ implies that

$$E(x, y) * \underbrace{\tau(x \leqslant z)}_{=1} \leq E(z, y)\,.$$

Hence, $\alpha(x, y) \leq \alpha(z, y)$.

Conversely, let α be an L-tolerance relation on X with $\alpha \subseteq \triangledown^c \cup \delta$ such that α is increasing on $\triangledown^c$ and $E = \triangledown \cup \alpha$. Since $\triangledown$ and α are L-tolerance relations on X, it holds indeed that E is an L-tolerance relation on X. It remains to show that E is compatible with $\leqslant$, i.e.,

$$E(x, y) * \tau(x \leqslant z) \leq E(z, y)\,,$$

for any $x, y, z \in X$. We consider the following cases:

(i) The case $x \triangledown y$, i.e., $E(x, y) = \tau(x \triangledown y) = 1$. It then follows from the compatibility of $\triangledown$ with $\leqslant$ (see Proposition 5) that

$$\begin{aligned} E(x, y) * \tau(x \leqslant z) &= \tau(x \triangledown y) * \tau(x \leqslant z) \\ &\leq \tau(z \triangledown y) \\ &\leq E(z, y)\,. \end{aligned}$$

(ii) The case $x \triangledown^c y$, i.e., $E(x, y) = \alpha(x, y)$. We consider two subcases:

(a) The case $z \triangledown y$, i.e., $E(z, y) = \tau(z \triangledown y) = 1$. It then trivially holds that

$$E(x, y) * \tau(x \leqslant z) \leq E(z, y) = 1\,.$$

(b) The case $z \triangledown^c y$, i.e., $E(z,y) = \alpha(z,y)$. If $\neg(x \leqslant z)$, then it trivially holds that

$$E(x,y) * \tau(x \leqslant z) = 0 \leq E(z,y)\,.$$

If $x \leqslant z$, then the fact that α is increasing on $\triangledown^c$ implies that

$$\begin{aligned} E(x,y) * \tau(x \leqslant z) &= \alpha(x,y) * \tau(x \leqslant z) \\ &\leq \alpha(z,y) = E(z,y)\,. \quad \square \end{aligned}$$

Example 3. Consider the set of positive integers $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$, which can be turned into a poset when equipped with the divisibility relation $|$ (i.e. $x|y$ means that x is a divisor of y, which implies in particular that x is smaller than or equal to y). Obviously, it holds that $\triangledown = \left(\mathbb{N}^+\right)^2$ as 1 is a trivial divisor of any $n \in \mathbb{N}^+$. Corollary 1 then implies that the only L-tolerance relation E that is compatible with $|$ is given by $E = X^2$.

Next, let us consider $\mathbb{N}^{>1} = \mathbb{N} \setminus \{0, 1\}$, again equipped with the divisibility relation $|$. Now the relation $\triangledown$ is given by

$$\triangledown = \{(x,y) \in \left(\mathbb{N}^{>1}\right)^2 \mid x \text{ and } y \text{ have a common divisor} > 1\}\,.$$

Let $L = [0, 1]$ and consider an increasing function $g : \left(\mathbb{N}^{>1}\right)^2 \to \mathbb{R}^+$ (such as addition or multiplication), then Theorem 1 implies that any fuzzy tolerance relation

$$E(x,y) = \begin{cases} 1 & \text{, if } x \text{ and } y \text{ have a common divisor in } \mathbb{N}^{>1} \\ 1 - e^{-g(x,y)} & \text{, otherwise} \end{cases}$$

is compatible with the divisibility relation $|$.

3.3. Compatibility of an L-equivalence relation with an order relation

In this subsection, we provide a characterization of the L-equivalence relations that are compatible with a given order relation.

Theorem 2. *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. Then the following statements are equivalent:*

(i) *E is compatible with $\leqslant$;*
(ii) *$\leqslant \subseteq E$;*
(iii) *$\triangledown \subseteq E$;*
(iv) *$\vartriangle \subseteq E$;*
(v) *$\boxtimes \subseteq E$.*

Proof. (i) ⇒ (ii): Since E is compatible with $\leqslant$, it follows from Proposition 6 that $\triangledown \subseteq E$. Since $\leqslant \subseteq \triangledown$, it holds that $\leqslant \subseteq E$.

(ii) ⇒ (iii): Let $x, y \in X$ such that $x \triangledown y$, then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. Since $\leqslant \subseteq E$, it follows that $E(c,x) = E(c,y) = 1$. The symmetry and $*$-transitivity of E then imply that

$$\underbrace{E(c,x)}_{=1} * \underbrace{E(c,y)}_{=1} = E(x,c) * E(c,y) \leq E(x,y)\,.$$

Hence, $E(x,y) = 1$. Thus, $\triangledown \subseteq E$.

(iii) ⇒ (iv): Let $x, y \in X$ such that $x \vartriangle y$, then there exists $c \in X$ such that $x \leqslant c$ and $y \leqslant c$. This implies that $x \triangledown c$ and $y \triangledown c$. Since $\triangledown \subseteq E$, it follows that $E(x,c) = E(y,c) = 1$. The symmetry and $*$-transitivity of E then imply that

$$\underbrace{E(x,c)}_{=1} * \underbrace{E(y,c)}_{=1} = E(x,c) * E(c,y) \leq E(x,y)\,.$$

Hence, $E(x,y) = 1$. Thus, $\vartriangle \subseteq E$.

(iv) ⇒ (v): Let $x, y \in X$ such that $x \boxtimes y$, then it holds that $x \triangledown y$ or $x \vartriangle y$.

(a) If $x \vartriangle y$, then from the hypothesis $\vartriangle \subseteq E$ it follows that $E(x, y) = 1$.
(b) If $x \triangledown y$, then there exists $c \in X$ such that $c \leqslant x$ and $c \leqslant y$. This implies that $x \vartriangle c$ and $y \vartriangle c$. Since $\vartriangle \subseteq E$, it follows that $E(x, c) = E(y, c) = 1$. The symmetry and $*$-transitivity of E then imply that

$$\underbrace{E(x, c)}_{=1} * \underbrace{E(y, c)}_{=1} = E(x, c) * E(c, y) \leq E(x, y)\,.$$

Hence, $E(x, y) = 1$.

We conclude that $\boxtimes \subseteq E$.
(v) ⇒ (i): Since $\leqslant \subseteq \boxtimes \subseteq E$, it follows that

$$E(x, y) * \tau(x \leqslant z) \leq E(x, y) * E(x, z)\,.$$

The symmetry and $*$-transitivity of E then imply that

$$E(x, y) * \tau(x \leqslant z) \leq E(z, y)\,.$$

Thus, E is compatible with $\leqslant$. □

In view of Proposition 4, we obtain the following corollary. In addition to the case of L-tolerance relations, it shows that there exist several other mild conditions under which the compatibility of an L-equivalence relation with an order relation is a trivial notion.

Corollary 2. *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. If at least one of the following conditions is satisfied:*

(i) *$(X, \leqslant)$ is a $\wedge$-semi lattice,*
(ii) *$(X, \leqslant)$ is a $\vee$-semi lattice,*
(iii) *$(X, \leqslant)$ has a smallest element,*
(iv) *$(X, \leqslant)$ has a greatest element,*

then it holds that E is compatible with $\leqslant$ if and only if $E = X^2$.

For a relation R on a set X, let R^* denote its transitive closure, i.e., the smallest transitive relation on X that contains R. The transitive closure of a reflexive (resp. symmetric) relation is reflexive (resp. symmetric) as well.

As a consequence of Theorem 2, the following proposition shows that the transitive closures $\triangledown^*$, $\vartriangle^*$ and $\boxtimes^*$ coincide and are compatible with $\leqslant$.

Proposition 7. *Let $(X, \leqslant)$ be a poset, then it holds that the transitive closures $\triangledown^*$, $\vartriangle^*$ and $\boxtimes^*$ coincide and are compatible with $\leqslant$.*

Proof. First, note that the transitive closures $\triangledown^*$, $\vartriangle^*$ and $\boxtimes^*$ are equivalence relations on X. Next, we show that $\triangledown^* = \boxtimes^*$. Since $\triangledown \subseteq \triangledown^*$, it follows from Theorem 2 that $\triangledown^*$ is compatible with $\leqslant$. Again, from Theorem 2 it follows that $\boxtimes \subseteq \triangledown^*$. The fact that $\boxtimes^*$ is the smallest transitive relation containing $\boxtimes$ then implies that $\boxtimes^* \subseteq \triangledown^*$. Conversely, since $\triangledown \subseteq \boxtimes$, it holds that $\triangledown^* \subseteq \boxtimes^*$. Thus, $\triangledown^* = \boxtimes^*$. In a similar way, we show that $\vartriangle^* = \boxtimes^*$. Finally, Theorem 2 guarantees that the relation $E = \triangledown^* = \vartriangle^* = \boxtimes^*$ is compatible with $\leqslant$. □

4. Left- and right-compatibility of an order relation with an L-tolerance relation

In [17], we studied the compatibility of an order relation with a reflexive L-relation. The compatibility of an order relation $\leqslant$ on X with an L-relation R on X states that:

$$\tau(x \leqslant y) * R(x,z) * R(y,t) \leq \tau(z \leqslant t)\,, \tag{6}$$

for any $x, y, z, t \in X$. It turns out that the crisp equality is the only reflexive L-relation that the order relation $\leqslant$ is compatible with. This result is easily generalized to the left- and right-compatibility of $\leqslant$ with any L-relation R.

Theorem 3. *Let $(X, \leqslant)$ be a poset and R be an L-relation on X. Then it holds that*

(i) *$\leqslant$ is left-compatible with R if and only if $R \subseteq \leqslant^t$;*
(ii) *$\leqslant$ is right-compatible with R if and only if $R \subseteq \leqslant$.*

Proof. (i) Suppose that $\leqslant$ is left-compatible with R, i.e., $\tau(x \leqslant y) * R(x,z) \leq \tau(z \leqslant y)$, for any $x, y, z \in X$. Since $\leqslant$ is reflexive, it holds in particular that

$$\underbrace{\tau(x \leqslant x)}_{=1} * R(x,y) \leq \tau(y \leqslant x)\,,$$

for any $x, y \in X$, i.e., $R(x,y) \leq \tau(y \leqslant x)$. Hence, $R(x,y) \leq \tau(x \leqslant^t y)$. Thus, $R \subseteq \leqslant^t$.
Conversely, if $R \subseteq \leqslant^t$, then it follows that

$$\begin{aligned}\tau(x \leqslant y) * R(x,z) &\leq \tau(x \leqslant y) * \tau(x \leqslant^t z)\\ &\leq \tau(x \leqslant y) * \tau(z \leqslant x)\\ &\leq \tau(z \leqslant y)\,,\end{aligned}$$

for any $x, y, z \in X$. Hence, $\leqslant$ is left-compatible with R.
(ii) Follows from Lemma 1 and (i). □

Corollary 3. *Let $(X, \leqslant)$ be a poset and R be an L-relation on X. Then it holds that $\leqslant$ is left- and right-compatible with R if and only if $R \subseteq \delta$.*

Combining Proposition 1 and Corollary 3 easily leads to the conclusion that $\leqslant$ is compatible with a reflexive L-relation R if and only if R is the crisp equality on X, which corresponds to Theorem 1 in [11].

Also, using the fact that the crisp equality δ is the only L-tolerance relation on X that satisfies $\delta \subseteq \leqslant$, the following corollary is obvious.

Corollary 4. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. Then it holds that $\leqslant$ is left- or right-compatible with E if and only if E is the crisp equality on X.*

This corollary confirms that, just as for compatibility, left- and right-compatibility of an order relation with an L-tolerance relation are void concepts. For this reason, we turn our attention to strict order relations in the next section.

5. Left- and right-compatibility of a strict order relation with an L-tolerance or L-equivalence relation

In this section, we establish a representation of the L-tolerance and L-equivalence relations that a given strict order is left- or right-compatible with. We pay particular attention to the rectification of the representation theorems (Theorem 2–4 and their corollaries) stated in [11].

5.1. Left-clone and right-clone relations of a poset

In this subsection, we generalize the notion of clone relation of a poset $(X, \leqslant)$ introduced in [11] to the notions of left-clone and right-clone relation.

Definition 3. [11] The clone relation $\approx$ of a poset $(X, \leqslant)$ is the binary relation on X defined by

$$x \approx y \quad \text{if and only if} \quad \begin{cases} (\forall z \in X \setminus \{x, y\})(z < x \Leftrightarrow z < y) \\ \text{and} \\ (\forall z \in X \setminus \{x, y\})(x < z \Leftrightarrow y < z)\,. \end{cases} \tag{7}$$

Definition 4.

(i) The left-clone relation $\approx^\ell$ of a poset $(X, \leqslant)$ is the binary relation on X defined by

$$x \approx^\ell y \quad \text{if and only if} \quad (\forall z \in X \setminus \{x, y\})(x < z \Leftrightarrow y < z)\,. \tag{8}$$

(ii) The right-clone relation $\approx^r$ of a poset $(X, \leqslant)$ is the binary relation on X defined by

$$x \approx^r y \quad \text{if and only if} \quad (\forall z \in X \setminus \{x, y\})(z < x \Leftrightarrow z < y)\,. \tag{9}$$

Given the clone, left-clone and right-clone relations of a poset $(X, \leqslant)$, we consider the following binary relations on X:

$$\begin{aligned} \diamond &= \approx \cap \parallel = \{(x, y) \in X^2 \mid x \approx y \ \wedge \ x \parallel y\} \\ \diamond^\ell &= \approx^\ell \cap \parallel = \{(x, y) \in X^2 \mid x \approx^\ell y \ \wedge \ x \parallel y\} \\ \diamond^r &= \approx^r \cap \parallel = \{(x, y) \in X^2 \mid x \approx^r y \ \wedge \ x \parallel y\}\,. \end{aligned}$$

Obviously, it holds that $\approx = \approx^\ell \cap \approx^r$ and $\diamond = \diamond^\ell \cap \diamond^r$. If $x \diamond y$, then we say that x and y are incomparable clones, and if $x \diamond^\ell y$ (resp. $x \diamond^r y$), then we say that x and y are incomparable left-clones (resp. right-clones).

The following proposition is immediate.

Proposition 8. *Let $(X, \leqslant)$ be a poset with corresponding $\approx$, $\approx^\ell$ and $\approx^r$. Consider the poset $(X, \leqslant^t)$ with corresponding $\approx_t$, $\approx_t^\ell$ and $\approx_t^r$, then it holds that $\approx = \approx_t$, $\approx^\ell = \approx_t^r$ and $\diamond^\ell = \diamond_t^r = \approx_t^r \cap \parallel$.*

The following proposition lists all cases in which $x \not\approx^\ell y$ (resp. $x \not\approx^r y$).

Proposition 9. *Let $(X, \leqslant)$ be a poset with corresponding relations $\approx^\ell$ and $\approx^r$. For any $x, y \in X$, it holds that*

(a) *$x \not\approx^\ell y$ if and only if there exists $z \in X \setminus \{x, y\}$ such that exactly one of the following statements holds:*
   (i) *$x < z < y$ or $y < z < x$;*
   (ii) *$x \ll y$, $x < z$ and $y \parallel z$;*
   (iii) *$y \ll x$, $y < z$ and $x \parallel z$;*
   (iv) *$x \parallel y$, $x < z$ and $y \parallel z$;*
   (v) *$x \parallel y$, $y < z$ and $x \parallel z$.*

(b) *$x \not\approx^r y$ if and only if there exists $z \in X \setminus \{x, y\}$ such that exactly one of the following statements holds:*
   (i) *$x < z < y$ or $y < z < x$;*
   (ii) *$x \ll y$, $z < y$ and $x \parallel z$;*
   (iii) *$y \ll x$, $z < x$ and $y \parallel z$;*
   (iv) *$x \parallel y$, $z < x$ and $y \parallel z$;*
   (v) *$x \parallel y$, $z < y$ and $x \parallel z$.*

The following proposition discusses the irreflexivity, symmetry and transitivity of the relations $\diamond^\ell$, $\diamond^r$ and $\diamond$.

Proposition 10. *Let $(X, \leqslant)$ be a poset. The corresponding relations $\diamond^\ell$, $\diamond^r$ and $\diamond$ are irreflexive, symmetric and transitive.*

Proof. Since $\diamond^\ell$, $\diamond^r$ and $\diamond$ are obviously irreflexive and symmetric, it only remains to show that they are transitive. Let $x, y, z \in X$.

(i) Suppose that $x \diamond^\ell y$ and $y \diamond^\ell z$, i.e., ($x \approx^\ell y$ and $x \parallel y$) and ($y \approx^\ell z$ and $y \parallel z$). We will show that $x \approx^\ell z$ and $x \parallel z$. Assume that $x \not\approx^\ell z$ or $x \nparallel z$.
   (a) If $x \nparallel z$, then $x < z$ or $z < x$. Since $x \approx^\ell y$ and $y \approx^\ell z$, it follows that $y < z$ or $y < x$, whence $y \nparallel z$ or $x \nparallel y$, a contradiction. Therefore, $x \parallel z$.
   (b) If $x \not\approx^\ell z$, then from Proposition 9 and the fact that $x \parallel z$, it follows that there exists $w \in X \setminus \{x, z\}$ such that ($x < w$ and $z \parallel w$) or ($z < w$ and $x \parallel w$). Since $x \approx^\ell y$ and $y \approx^\ell z$, it follows that ($y < w$ and $z \parallel w$) or ($y < w$ and $x \parallel w$). From Proposition 9, it follows that $x \not\approx^\ell y$ or $y \not\approx^\ell z$, a contradiction. Hence, $\diamond^\ell$ is transitive.

(ii) Follows immediately from (i) and Proposition 8.
(iii) Follows from the fact that $\diamond = \diamond^\ell \cap \diamond^r$. □

In what follows, we use the following brief notations:

$$\diamond_= = \diamond \cup \delta$$
$$\diamond^\ell_= = \diamond^\ell \cup \delta$$
$$\diamond^r_= = \diamond^r \cup \delta\,.$$

Corollary 5. *The relations $\diamond^\ell_=$, $\diamond^r_=$ and $\diamond_=$ associated with a poset $(X, \leq)$ are equivalence relations on X.*

Example 4. Consider the set of positive integers $\mathbb{N}^+$ equipped with the divisibility relation $|$; let $\mathbb{P}$ denote the set of prime numbers. One easily verifies that the left-clone relation $\approx^\ell$ is given by δ (and, hence, also the clone relation $\approx$ is given by δ). Further, two different positive integers x and y are right-clones if and only if one of the following statements holds:

(i) $(x, y) \in \mathbb{P}^2$ (and, thus, they are incomparable);
(ii) there exist $p \in \mathbb{P}$ and $n \in \mathbb{N}^+$ such that $(x, y) = (p^n, p^{n+1})$ or $(x, y) = (p^{n+1}, p^n)$ (and, thus, they are comparable).

Example 5. Consider the set of human beings, which can be turned into a poset when equipped with the ancestor relation (where $x \leqslant y$ means that y descends from x). One easily verifies that two different individuals are left-clones if and only if one of the following statements holds:

(i) they have the same offspring (and, thus, they are incomparable);
(ii) one of them is the only child of the other one (and, thus, they are comparable).

Similarly, one can verify that two different individuals are right-clones if and only if one of the following statements holds:

(i) they are siblings (and, thus, they are incomparable);
(ii) the first (resp. second) is a child of the second (resp. first), while the other parent of the first (resp. second) is an ancestor of the second (resp. first) (and, thus, they are comparable).

5.2. Left- and right-compatibility of a strict order relation with an L-tolerance relation

In this subsection, we study the left- and right-compatibility of a given strict order relation with an L-tolerance relation. First, we show the following key results.

Proposition 11. *Let $<$ be the strict order relation of a poset $(X, \leqslant)$. Then it holds that*

(i) *$<$ is left-compatible with the relation $\diamond^\ell_=$;*
(ii) *$<$ is right-compatible with the relation $\diamond^r_=$.*

Proof. (i) Let $x, y, z \in X$. Since $\diamond^\ell = \approx^\ell \cap \parallel$, it follows that ($x < y$ and $x \diamond^\ell z$) implies $z < y$. Hence,

$$\tau(x < y) * \tau(x \diamond^\ell z) \leq \tau(z < y)\,.$$

Thus, $<$ is left-compatible with $\diamond^\ell$, and obviously also with $\diamond^\ell_=$.

(ii) Follows immediately from (i), Lemma 1 and Proposition 8. □

Corollary 6. *The strict order relation $<$ of a poset $(X, \leqslant)$ is compatible with the relation $\diamond_=$.*

Proof. Proposition 11 states that $<$ is left-compatible with the relation $\diamond^\ell_=$ and right-compatible with the relation $\diamond^r_=$. Let $x, y, z, t \in X$, then it follows that

$$\begin{aligned}\tau(x < y) * \tau(x \diamond_= z) * \tau(y \diamond_= t) &= \tau(x < y) * \tau(x(\diamond^\ell_= \cap \diamond^r_=)z) * \tau(y(\diamond^\ell_= \cap \diamond^r_=)t)\\ &\leq \tau(x < y) * \tau(x \diamond^\ell_= z) * \tau(y \diamond^r_= t)\\ &\leq \tau(z < y) * \tau(y \diamond^r_= t)\\ &\leq \tau(z < t)\,.\end{aligned}$$

Hence, $<$ is compatible with the relation $\diamond_=$. □

The following theorem characterizes the left-compatibility (resp. right-compatibility) of the strict order relation $<$ of a poset $(X, \leqslant)$ with an L-tolerance relation in terms of the relation $\diamond^\ell_=$ (resp. $\diamond^r_=$).

Theorem 4. *Let $<$ be the strict order relation of a poset $(X, \leqslant)$ and E be an L-tolerance relation on X. Then it holds that*

(i) *$<$ is left-compatible with E if and only if $E \subseteq \diamond^\ell_=$;*
(ii) *$<$ is right-compatible with E if and only if $E \subseteq \diamond^r_=$.*

Proof. (i) Suppose that $<$ is left-compatible with E and let $x, y \in X$. If $E(x, y) = 0$, then it trivially holds that $E(x, y) \leq \tau(x \diamond^\ell_= y)$. If $E(x, y) > 0$, then we will show that $\tau(x \diamond^\ell_= y) = 1$, i.e., $x \diamond^\ell y$ or $x = y$. First, we will show that $x = y$ or $x \parallel y$. Suppose that $x < y$, then the left-compatibility of $<$ with E implies that

$$0 < E(x, y) = \underbrace{\tau(x < y)}_{=1} * E(x, y) \leq \tau(y < y)\,.$$

This implies that $\tau(y < y) > 0$, i.e., $\tau(y < y) = 1$, a contradiction. Suppose that $y < x$, then the symmetry of E and the left-compatibility of $<$ with E imply that

$$0 < E(x, y) = E(y, x) = \underbrace{\tau(y < x)}_{=1} * E(y, x) \leq \tau(x < x)\,.$$

This implies that $\tau(x < x) > 0$, i.e., $\tau(x < x) = 1$, a contradiction. Thus, $x = y$ or $x \parallel y$. Next, we show that $x \approx^\ell y$. Let $z \in X \setminus \{x, y\}$. We consider two cases:

(a) If $x < z$, then the left-compatibility of $<$ with E implies that

$$0 < E(x, y) = \underbrace{\tau(x < z)}_{=1} * E(x, y) \leq \tau(y < z)\,.$$

Hence, $\tau(y < z) > 0$, i.e., $y < z$.

(b) If $y < z$, then the symmetry of E and the left-compatibility of $<$ with E imply that

$$0 < E(x, y) = \underbrace{\tau(y < z)}_{=1} * E(y, x) \leq \tau(x < z)\,.$$

Hence, $\tau(x < z) > 0$, i.e., $x < z$.

We conclude that if $E(x, y) > 0$, then it holds that ($x = y$ or $x \parallel y$) and $x \approx^\ell y$. Hence, $x \diamond^\ell_= y$. Thus, $E \subseteq \diamond^\ell_=$. Conversely, if $E \subseteq \diamond^\ell_=$, then due to the monotonicity of $*$, it holds that

$$\tau(x < y) * E(x, z) \leq \tau(x < y) * \tau(x \diamond^\ell_= z),$$

for any $x, y, z \in X$. From Proposition 11 (i), it follows that

$$\tau(x < y) * E(x, z) \leq \tau(z < y),$$

for any $x, y, z \in X$. Thus, $<$ is left-compatible with E.

(ii) Follows from (i), Lemma 1 and Proposition 8. □

The above theorem implies that the relation $\diamond^\ell_=$ (resp. $\diamond^r_=$) is the greatest L-tolerance relation on X such that $<$ is left-compatible (resp. right-compatible) with it.

Combining Theorem 4 and Proposition 1 leads to the following corollary. We stress that this corollary rectifies Theorem 2 in [11] claiming that the compatibility of the strict order relation of a poset with an L-tolerance relation is equivalent to the inclusion of that L-tolerance relation in the clone relation of that poset. More precisely, the rectification is that $<$ is compatible with E if and only if $E \subseteq \diamond_=$ (and not the inclusion $E \subseteq \approx$).

Corollary 7. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-tolerance relation on X. Then it holds that $<$ is compatible with E if and only if $E \subseteq \diamond_=$.*

Corollaries 6 and 7 imply that the relation $\diamond_=$ (resp. the crisp equality) is the greatest (resp. smallest) L-tolerance relation on X such that $<$ is compatible with it.

Theorem 4 leads to the following representation theorem of all L-tolerance relations a given strict order relation is left- or right-compatible with.

Theorem 5. *Let $(X, \leqslant)$ be a poset and E be an L-tolerance relation on X. Then it holds that*

(i) *$<$ is left-compatible with E if and only if there exists an L-tolerance relation α on X with $\alpha \subseteq \diamond^\ell_=$ such that*

$$E(x, y) = \begin{cases} 1 & , \text{if } x = y \\ \alpha(x, y) & , \text{if } x \diamond^\ell y \\ 0 & , \text{otherwise} \end{cases}$$

(ii) *$<$ is right-compatible with E if and only if there exists an L-tolerance relation α on X with $\alpha \subseteq \diamond^r_=$ such that*

$$E(x, y) = \begin{cases} 1 & , \text{if } x = y \\ \alpha(x, y) & , \text{if } x \diamond^r y \\ 0 & , \text{otherwise} \end{cases}$$

Proof. (i) Suppose that $<$ is left-compatible with E. We consider the L-relation α on X defined as

$$\alpha(x, y) = \begin{cases} E(x, y) & , \text{if } x \diamond^\ell_= y \\ 0 & , \text{otherwise} \end{cases}$$

Since the relations $\diamond^\ell_=$ and E are symmetric, it holds that α is an L-tolerance relation on X. Since $<$ is left-compatible with E, it follows from Theorem 4 that $E \subseteq \diamond^\ell_=$. Thus,

$$E(x, y) = \begin{cases} 1 & , \text{if } x = y \\ \alpha(x, y) & , \text{if } x \diamond^\ell y \\ 0 & , \text{otherwise} \end{cases}$$

Conversely, let $\alpha \subseteq \diamond^{\ell}_{=}$ be an L-tolerance relation on X. It then follows that the L-relation E defined as

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond^{\ell} y\\ 0 & \text{, otherwise}\end{cases}$$

is an L-tolerance relation on X. Also, since $E \subseteq \diamond^{\ell}_{=}$, Theorem 4(i) guarantees that $<$ is left-compatible with E.

(ii) Follows from (i), Lemma 1 and Proposition 8. □

Combining Theorem 5 and Proposition 1 leads to the following corollary. We stress that this corollary rectifies the representation theorem (Theorem 3) in [11] of the L-tolerance relations a strict order relation is compatible with.

Corollary 8. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-tolerance relation on X. Then it holds that $<$ is compatible with E if and only if there exists an L-tolerance relation α on X with $\alpha \subseteq \diamond_{=}$ such that*

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond y\\ 0 & \text{, otherwise}\end{cases}$$

Corollary 9. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be a crisp tolerance relation on X. Then it holds that $<$ is compatible with E if and only if there exists a crisp tolerance relation α on X with $\alpha \subseteq \diamond_{=}$ such that*

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond y\\ 0 & \text{, otherwise}\end{cases}$$

Example 6. Consider the set of positive integers $\mathbb{N}^+$ equipped with the divisibility relation $|$; let $\mathbb{P}$ denote the set of prime numbers. Let $|_{\neq}$ denote the strict part of the divisibility relation. For any L-tolerance relation E on $\mathbb{N}^+$, the following statements easily follow using Example 4:

(i) $|_{\neq}$ is (left-)compatible with E if and only if $E=\delta$;

(ii) $|_{\neq}$ is right-compatible with E if and only if there exists an L-tolerance relation α on $\mathbb{P}$ such that

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } (x,y)\in\mathbb{P}^2\\ 0 & \text{, otherwise}\end{cases}$$

5.3. Left- and right-compatibility of a strict order relation with an L-equivalence relation

In this subsection, we study the left- and right-compatibility of a given strict order relation with an L-equivalence relation.

Theorem 6. *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. Then it holds that*

(i) *$<$ is left-compatible with E if and only if there exists an L-equivalence relation α on X with $\alpha \subseteq \diamond^{\ell}_{=}$ such that*

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond^{\ell} y\\ 0 & \text{, otherwise}\end{cases}$$

(ii) *$<$ is right-compatible with E if and only if there exists an L-equivalence relation α on X with $\alpha \subseteq \diamond^r_=$ such that*

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond^r y\\ 0 & \text{, otherwise}\end{cases}$$

Proof. (i) Theorem 5 states that $<$ is left-compatible with E if and only if there exists an L-tolerance relation α on X with $\alpha \subseteq \diamond^\ell_=$ such that

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond^\ell y\\ 0 & \text{, otherwise}\end{cases}$$

where

$$\alpha(x,y)=\begin{cases}E(x,y) & \text{, if } x\diamond^\ell_= y\\ 0 & \text{, otherwise}\end{cases}$$

It remains to show that α is $*$-transitive, i.e.,

$$\alpha(x,y)*\alpha(y,z)\le\alpha(x,z)\,,$$

for any $x, y, z \in X$. We consider the following cases:

(a) If $x = y$ or $y = z$, then it trivially holds that $\alpha(x, y) * \alpha(y, z) \le \alpha(x, z)$.

(b) If $x \diamond^\ell y$ and $y \diamond^\ell z$, then $\alpha(x, y) = E(x, y)$ and $\alpha(y, z) = E(y, z)$. The transitivity of $\diamond^\ell$ implies that $\alpha(x, z) = E(x, z)$. Since E is $*$-transitive, it follows that $E(x, y) * E(y, z) \le E(x, z)$, i.e., $\alpha(x, y) * \alpha(y, z) \le \alpha(x, z)$.

(c) If $(x, y) \notin \diamond^\ell_=$ or $(y, z) \notin \diamond^\ell_=$, then $\alpha(x, y) = 0$ or $\alpha(y, z) = 0$. Hence, it trivially holds that $\alpha(x, y)*\alpha(y, z) \le \alpha(x, z)$.

We conclude that α is $*$-transitive. Thus, α is an L-equivalence relation on X.

Conversely, since α is an L-equivalence relation on X and $\diamond^\ell$ is transitive, it obviously holds that the L-relation E defined as

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond^\ell y\\ 0 & \text{, otherwise}\end{cases}$$

is an L-equivalence relation on X.

(ii) Follows from (i), Lemma 1 and Proposition 8. □

As a corollary, we obtain the following representation of the L-equivalence relations that a given strict order relation is compatible with. This result rectifies our result in ([11], Theorem 4).

Corollary 10. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-equivalence relation on X. Then it holds that $<$ is compatible with E if and only if there exist an L-equivalence relation α on X with $\alpha \subseteq \diamond_=$ such that*

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond y\\ 0 & \text{, otherwise}\end{cases}$$

Corollary 11. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-equality relation on X. Then it holds that $<$ is compatible with E if and only if there exist an L-equality relation α on X with $\alpha \subseteq \diamond_=$ such that*

$$E(x,y)=\begin{cases}1 & \text{, if } x=y\\ \alpha(x,y) & \text{, if } x\diamond y\\ 0 & \text{, otherwise}\end{cases}$$

Corollary 12. *Let* $(X, \leqslant)$ *be a poset with clone relation* $\approx$ *and* E *be a crisp equivalence relation on* X. *Then it holds that* $<$ *is compatible with* E *if and only if there exists a crisp equivalence relation* α *on* X *with* $\alpha \subseteq \diamond_=$ *such that*

$$E(x, y) = \begin{cases} 1 & , \text{ if } x = y \\ \alpha(x, y) & , \text{ if } x \diamond y \\ 0 & , \text{ otherwise} \end{cases}$$

6. Conclusion

In this work, we have extended the study of the compatibility of (strict) order relations with fuzzy tolerance relations initiated by De Baets et al. [11] to the left- and right-compatibility of (strict) order relations and fuzzy tolerance relations. We have characterized the fuzzy tolerance relations that are compatible with a given (strict) order relation. Conversely, we have provided a representation of the fuzzy tolerance relations a given strict order is left- or right-compatible with. Specific attention has been paid to the case of fuzzy equivalence relations. These results were obtained through the use of the newly introduced notions of left-clone and right-clone relations associated with a given order relation. We believe that these notions, or in general, the left-clone and right-clone relations associated with an arbitrary crisp or fuzzy relation are worthy of further investigation.

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