

ON THE COMPATIBILITY OF A CRISP RELATION WITH A FUZZY EQUIVALENCE RELATION

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This paper is dedicated to Professor L. A. Zadeh on the occasion of his 95th birthday and the 50th year of the birth of fuzzy logic

ABSTRACT. In a recent paper, De Baets et al. have characterized the fuzzy tolerance and fuzzy equivalence relations that a given strict order relation is compatible with. In this paper, we generalize this characterization by considering an arbitrary (crisp) relation instead of a strict order relation, while paying attention to the particular cases of a reflexive or irreflexive relation. The reasoning largely draws upon the notion of the clone relation of a (crisp) relation, introduced recently by Bouremel et al., and the partition of this clone relation in terms of three different types of pairs of clones. More specifically, reflexive related clones and irreflexive unrelated clones turn out to play a key role in the characterization of the fuzzy tolerance and fuzzy equivalence relations that a given (crisp) relation is compatible with.

1. Introduction

The notion of compatibility of a given fuzzy relation with another one, extensively studied in [16], establishes an interesting relation on the set of fuzzy relations. It generalizes the notion of extensionality, introduced by Höhle and Blanchard [15], or the equivalent notion of compatibility, as it was coined by Bělohlávek [2], of a fuzzy relation w.r.t. a fuzzy equality relation. This notion appears, among others, in the study of fuzzy lattices [1, 17, 23, 25], in the study of fuzzy functions [12, 18, 19, 20], in the study of fuzzy order relations [3, 5, 6, 13] and in the lattice-theoretic approach to concept lattices [2].

Given the importance of fuzzy tolerance and fuzzy equivalence relations in the theory and applications of fuzzy sets, it is not surprising that compatibility has mainly been studied for a given fuzzy relation with the mentioned types of fuzzy relations. In this context, some of the present authors have focused their attention on the case where the given fuzzy relation is simply a crisp (strict) order relation, leading to surprising negative results as well as interesting representation theorems [11]. These results were obtained thanks to the introduction of the notion of clone relation associated with a strict order relation. Recently, it has been shown how a clone relation can be associated with any crisp relation [7]. Informally, two elements are called clones if they are related in the same way w.r.t. every other

Invited Paper: September 2016

Key words and phrases: Crisp relation, Fuzzy relation, Clone relation, Compatibility, Tolerance relation, Equivalence relation.

element. This allows us to take a step further in this paper and aim at characterizing the fuzzy tolerance and fuzzy equivalence relations a given crisp relation is compatible with.

This paper is structured as follows. After recalling some basic definitions and properties in Section 2, in particular related to the clone relation of a crisp relation, we study two auxiliary relations associated with this clone relation in Section 3. These auxiliary relations respectively gather the reflexive related clones and the irreflexive unrelated clones. In Section 4, we study the compatibility of a given crisp relation with the latter auxiliary relations. The results are exploited in Section 5 to characterize the fuzzy tolerance and fuzzy equivalence relations a given crisp relation is compatible with. These characterizations turn out to be pleasingly elegant and insightful. Finally, we present some conclusions and discuss future research in Section 6.

2. Basic Concepts

This section serves an introductory purpose. First, we recall some basic concepts and properties of binary relations and of L -relations. Second, we recall the notion of clone relation and its properties.

2.1. Binary Relations. A crisp relation (relation, for short) R on a set X is a subset of X^2 , i.e., a set of couples $(x, y) \in X^2$. We usually write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called unrelated elements, denoted by $x \parallel_R y$, if it neither holds that xRy nor that yRx . Otherwise, they are called related elements, denoted by $x \not\parallel_R y$. When no confusion can occur, we will write $x \parallel y$ or $x \not\parallel y$. We denote by R^c the *complement* of the relation R on X , i.e., for any $x, y \in X$, $xR^c y$ denotes the fact that $(x, y) \notin R$. We denote by R^t the *transpose* (*converse*) of the relation R on X , i.e., for any $x, y \in X$, $xR^t y$ denotes the fact that yRx . Finally, we denote by R^d the *dual* of the relation R on X , i.e., for any $x, y \in X$, $xR^d y$ denotes the fact that $yR^c x$.

A binary relation R on a set X is called:

- (i) *reflexive*: xRx , for any $x \in X$;
- (ii) *irreflexive*: $xR^c x$, for any $x \in X$;
- (iii) *symmetric*: xRy implies that yRx , for any $x, y \in X$;
- (iv) *antisymmetric*: xRy and yRx imply that $x = y$, for any $x, y \in X$;
- (v) *transitive*: xRy and yRz imply that xRz , for any $x, y, z \in X$;
- (vi) *antitransitive*: xRy and yRz imply that $xR^c z$, for any $x, y, z \in X$.

A relation R on a set X is called a *pseudo-order relation* if it is reflexive and antisymmetric [22]. A transitive pseudo-order relation \leq on a set X is called an *order relation*. A set X equipped with an order relation \leq is called a partially ordered set (poset, for short), denoted by (X, \leq) . For more details on relations, we refer to [9, 21].

2.2. Lattices. A poset (L, \leq) (see, e.g., [9]) is called a lattice if any two elements x and y have a greatest lower bound, denoted $x \frown y$ and called the meet (infimum) of x and y , as well as a smallest upper bound, denoted $x \vee y$ and called the join (supremum) of x and y . A lattice can also be defined as an algebraic structure,

namely a set L equipped with two binary operations \frown and \smile that are idempotent, commutative and associative, and satisfy the absorption laws ($x \frown (x \smile y) = x$ and $x \smile (x \frown y) = x$, for any $x, y \in L$). The order relation and the meet and join operations are related as follows: $x \leq y$ if and only if $x \frown y = x$; $x \leq y$ if and only if $x \smile y = y$.

A bounded lattice is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$ for every x in L . Often, the notation $(L, \leq, \frown, \smile, 0, 1)$ is used.

2.3. Fuzzy Relations. The notion of an L -relation on a set X generalizes the classical notion of a relation by expressing degrees of relationship in some bounded lattice $(L, \leq, \frown, \smile, 0, 1)$ [14]. A binary L -relation (L -relation, for short) R on X is a mapping $R: X \times X \rightarrow L$. If $L = \{0, 1\}$, then relations are obtained.

A triangular norm (t-norm, for short) $*$ on a bounded lattice L (see, e.g., [10]) is a binary operation on L that is commutative (i.e., $\alpha * \beta = \beta * \alpha$, for any $\alpha, \beta \in L$) and associative (i.e., $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$, for any $\alpha, \beta, \gamma \in L$), has neutral element 1 (i.e., $\alpha * 1 = \alpha$, for any $\alpha \in L$) and is order-preserving (i.e., if $\alpha \leq \beta$, then $\alpha * \gamma \leq \beta * \gamma$, for any $\alpha, \beta, \gamma \in L$). Throughout this paper, L always denotes a bounded lattice $(L, \leq, \frown, \smile, 0, 1)$ and $*$ a t-norm on it.

The following properties of an L -relation R are of interest in this paper (see, e.g., [4, 8, 24]).

- (i) *reflexivity*: $R(x, x) = 1$, for any $x \in X$;
- (ii) *symmetry*: $R(x, y) = R(y, x)$, for any $x, y \in X$;
- (iii) **-transitivity*: $R(x, y) * R(y, z) \leq R(x, z)$, for any $x, y, z \in X$;
- (iv) *separability*: $R(x, y) = 1$ implies $x = y$, for any $x, y \in X$.

An L -relation R_1 is said to be included in an L -relation R_2 , denoted $R_1 \subseteq R_2$, if $R_1(x, y) \leq R_2(x, y)$, for any $x, y \in X$. The intersection of two L -relations R_1 and R_2 on X is the L -relation $R_1 \cap R_2$ on X defined by $R_1 \cap R_2(x, y) = R_1(x, y) \frown R_2(x, y)$, for any $x, y \in X$. Similarly, the union of two L -relations R_1 and R_2 on X is the L -relation $R_1 \cup R_2$ on X defined by $R_1 \cup R_2(x, y) = R_1(x, y) \smile R_2(x, y)$, for any $x, y \in X$. The transpose R^t of R is defined by $R^t(x, y) = R(y, x)$.

An L -relation R on X is called an L -tolerance relation if it is reflexive and symmetric. A $*$ -transitive L -tolerance relation is called an L -equivalence relation and a separable L -equivalence relation is called an L -equality relation. Note that the only $\{0, 1\}$ -equality relation on X is the smallest equivalence relation on X denoted by δ , i.e., $\delta = \{(x, y) \in X^2 \mid x = y\}$.

2.4. The Clone Relation of a Relation. In this subsection, we recall the notion of clone relation of a relation introduced by Bouremel et al. [7]. Two elements of a set equipped with a relation R are said to be a pair of clones if they are related in the same way w.r.t. every other element. More formally, the clone relation \approx_R of a relation R on a set X is the relation \approx_R on X defined by

$$x \approx_R y \quad \text{if} \quad \begin{cases} (\forall z \in X \setminus \{x, y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x, y\})(xRz \Leftrightarrow yRz). \end{cases}$$

Note that the clone relation \approx_R of a relation R on a set X is a tolerance relation on X . The clone relation \approx_R of a relation R can be partitioned as follows:

$$\approx_R = \triangleleft_R \cup \triangleright_R \cup \circ_R \cup \diamond_R \cup \delta,$$

where the relations \triangleleft_R , \triangleright_R , \diamond_R and \circ_R are given as:

$$\triangleleft_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge xRy \wedge yR^c x \wedge x \neq y\},$$

$$\triangleright_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge xR^c y \wedge yRx \wedge x \neq y\},$$

$$\circ_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge xRy \wedge yRx \wedge x \neq y\},$$

$$\diamond_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge x \parallel_R y \wedge x \neq y\}.$$

Note that, on the one hand, \triangleleft_R and \triangleright_R are irreflexive, antisymmetric and antitransitive and it holds that $\triangleleft_R = \triangleright_R^t$. On the other hand, \circ_R and \diamond_R are irreflexive, symmetric and transitive. Informally, \circ_R consists of ‘related clones’, while \diamond_R consists of ‘unrelated clones’.

An important special case is the clone relation of a strict order relation, which was introduced by De Baets et al. [11] and used for characterizing the L -tolerance and L -equivalence relations that a strict order relation is compatible with. For more details, we refer to [11, 16].

3. Two Auxiliary Relations

In this section, we study a subrelation of \circ_R and a subrelation of \diamond_R , associated with the clone relation \approx_R of a given relation R . These subrelations will turn out to be useful technical tools in the following sections.

Definition 3.1. Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R and \diamond_R . The following relations on X are defined:

- (i) $\circ_R^r = \{(x, y) \in \circ_R \mid xRx \wedge yRy\}$.
- (ii) $\diamond_R^i = \{(x, y) \in \diamond_R \mid xR^c x \wedge yR^c y\}$.

Obviously, \circ_R^r is a subrelation of \circ_R and \diamond_R^i is a subrelation of \diamond_R . Informally, \circ_R^r consists of ‘reflexive related clones’, while \diamond_R^i consists of ‘irreflexive unrelated clones’.

Remark 3.2. Definition 3.1 implies that, for a given relation R on a set X , the following two cases are impossible: $(x \circ_R^r y$ and $y \diamond_R^i z)$, as well as $(x \diamond_R^i y$ and $y \circ_R^r z)$.

Proposition 3.3. Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:

- (i) If R is reflexive, then $\circ_R^r = \circ_R$ and $\diamond_R^i = \emptyset$.
- (ii) If R is irreflexive, then $\diamond_R^i = \diamond_R$ and $\circ_R^r = \emptyset$.

Since \circ_R and \diamond_R are irreflexive and symmetric, the following proposition is immediate.

Proposition 3.4. Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:

- (i) The relation \circ_R^r is irreflexive and symmetric.

- (ii) The relation \diamond_R^i is irreflexive and symmetric.
- (iii) The relation $\circ_R^r \cup \diamond_R^i$ is irreflexive and symmetric.

Corollary 3.5. Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:

- (i) The relation $\circ_R^r \cup \delta$ is a tolerance relation.
- (ii) The relation $\diamond_R^i \cup \delta$ is a tolerance relation.
- (iii) The relation $\circ_R^r \cup \diamond_R^i \cup \delta$ is a tolerance relation.

The following proposition shows the relation between the subrelations associated with the clone relation \approx_R and the subrelations associated with the clone relations \approx_{R^c} , \approx_{R^t} and \approx_{R^d} . To that end, we need the following lemma.

Lemma 3.6. Let R be a relation on a set X . Then the following statements hold:

- (i) $\approx_{R^c} = \approx_R$.
- (ii) $\approx_{R^t} = \approx_R$.
- (iii) $\approx_{R^d} = \approx_R$.

Proof. (i) For any $x, y \in X$, it holds that

$$\begin{aligned} x \approx_{R^c} y &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x, y\})(zR^c x \Leftrightarrow zR^c y) \\ \text{and} \\ (\forall z \in X \setminus \{x, y\})(xR^c z \Leftrightarrow yR^c z) \end{cases} \\ &\Leftrightarrow \begin{cases} (\forall z \in X \setminus \{x, y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x, y\})(xRz \Leftrightarrow yRz) \end{cases} \\ &\Leftrightarrow x \approx_R y. \end{aligned}$$

- (ii) Is proved analogously to (i).
- (iii) Follows immediately from (i) and (ii). □

Proposition 3.7. Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:

- (i) $(\circ_{R^c}^r, \diamond_{R^c}^i) = (\diamond_R^i, \circ_R^r)$.
- (ii) $(\circ_{R^t}^r, \diamond_{R^t}^i) = (\circ_R^r, \diamond_R^i)$.
- (iii) $(\circ_{R^d}^r, \diamond_{R^d}^i) = (\diamond_R^i, \circ_R^r)$.

Proof. (i) We need to prove that $\circ_{R^c}^r = \diamond_R^i$ and $\diamond_{R^c}^i = \circ_R^r$.

(a) By definition, $\circ_{R^c}^r = \{(x, y) \in X^2 \mid (x \circ_{R^c} y) \wedge (xR^c x) \wedge (yR^c y)\}$. Since $\circ_{R^c} = \{(x, y) \in X^2 \mid x \approx_{R^c} y \wedge xR^c y \wedge yR^c x\}$ and $\approx_{R^c} = \approx_R$ (see Lemma 3.6), it follows that $\circ_{R^c}^r = \diamond_R^i$.

(b) By definition, $\diamond_{R^c}^i = \{(x, y) \in X^2 \mid (x \diamond_{R^c} y) \wedge (xRx) \wedge (yRy)\}$. Since $\diamond_{R^c} = \{(x, y) \in X^2 \mid (x \approx_{R^c} y) \wedge (xRy) \wedge (yRx)\}$ and $\approx_{R^c} = \approx_R$, it follows that $\diamond_{R^c}^i = \circ_R^r$.

- (ii) Is proved analogously to (i).
- (iii) Follows immediately from (i) and (ii). □

The following proposition identifies two important implications, which will be helpful in the proofs in Section 5.

Proposition 3.8. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:*

- (i) *If $x \circ_R^r y$, then, for any $z \in X \setminus \{x, y\}$, $x \approx_R z$ and zRz imply that $x \circ_R^r z$ and $y \circ_R^r z$.*
- (ii) *If $x \diamond_R^i y$, then, for any $z \in X \setminus \{x, y\}$, $x \approx_R z$ and $zR^c z$ imply that $x \diamond_R^i z$ and $y \diamond_R^i z$.*

Proof. (i) Let $x, y \in X$ and $z \in X \setminus \{x, y\}$ be such that $x \circ_R^r y$, $x \approx_R z$ and zRz . Note that $x \neq y$. On the one hand, since xRy , yRx , $x \approx_R z$, and $y \in X \setminus \{x, z\}$, it follows that zRy and yRz . On the other hand, since zRy , yRz , $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that zRx and xRz . As $x \approx_R z$, xRx and zRz , it follows that $x \circ_R^r z$.

Next, we prove that $y \circ_R^r z$. As yRy and zRz , it remains to prove that $y \approx_R z$. Suppose that $y \not\approx_R z$, then there exists $t \in X \setminus \{y, z\}$ such that $(yRt$ and $zR^c t)$ or $(zRt$ and $yR^c t)$ or $(tRy$ and $tR^c z)$ or $(tRz$ and $tR^c y)$. Suppose, for instance, that $(yRt$ and $zR^c t)$. Since zRx , it follows that $t \in X \setminus \{x, y, z\}$. It then holds that $x \approx_R y$ and $x \approx_R z$ imply that $(xRt$ and $xR^c t)$, a contradiction. The three other cases lead to a similar contradiction. Hence, $y \approx_R z$, and, therefore, $y \circ_R^r z$.

- (ii) Let $x, y \in X$ and $z \in X \setminus \{x, y\}$ be such that $x \diamond_R^i y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since $x \parallel_R y$, $x \approx_R z$ and $y \in X \setminus \{x, z\}$, it follows that $z \parallel_R y$. On the other hand, since $z \parallel_R y$, $x \approx_R y$ and $z \in X \setminus \{x, y\}$, it follows that $z \parallel_R x$. As $x \approx_R z$, $xR^c x$ and $zR^c z$, it follows that $x \diamond_R^i z$.

Next, we prove that $y \diamond_R^i z$. As $y \parallel_R z$, $yR^c y$ and $zR^c z$, it remains to prove that $y \approx_R z$. Suppose that $y \not\approx_R z$, then it follows that there exists $t \in X \setminus \{y, z\}$ such that $(yRt$ and $zR^c t)$ or $(zRt$ and $yR^c t)$ or $(tRy$ and $tR^c z)$ or $(tRz$ and $tR^c y)$. Suppose, for instance, that $(yRt$ and $zR^c t)$. Since $yR^c t$, it follows that $t \in X \setminus \{x, y, z\}$. It then holds that $x \approx_R y$ and $x \approx_R z$ imply that $(xRt$ and $xR^c t)$, a contradiction. The three other cases lead to a similar contradiction. Hence, $y \approx_R z$, and, therefore, $y \diamond_R^i z$. \square

The following proposition discusses the transitivity of the relations $\circ_R^r \cup \delta$, $\diamond_R^i \cup \delta$ and $\circ_R^r \cup \diamond_R^i \cup \delta$.

Proposition 3.9. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:*

- (i) *The relation $\circ_R^r \cup \delta$ is transitive.*
- (ii) *The relation $\diamond_R^i \cup \delta$ is transitive.*
- (iii) *The relation $\circ_R^r \cup \diamond_R^i \cup \delta$ is transitive.*

Proof. (i) Let $x, y, z \in X$ be such that $x(\circ_R^r \cup \delta)y$ and $y(\circ_R^r \cup \delta)z$.
 (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\circ_R^r \cup \delta)z$.
 (b) If $x \neq z$, $x \neq y$ and $y \neq z$, then it holds that $(x \circ_R^r y)$ and $(y \circ_R^r z)$. Since \circ_R^r is transitive, it holds that $x \circ_R^r z$. As xRx and zRz , it follows that $x \circ_R^r z$. Hence, $x(\circ_R^r \cup \delta)z$.

We conclude that $\circ_R^r \cup \delta$ is transitive.

- (ii) Let $x, y, z \in X$ be such that $x(\diamond_R^i \cup \delta)y$ and $y(\diamond_R^i \cup \delta)z$.
- (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\diamond_R^i \cup \delta)z$.
- (b) If $x \neq z$, $x \neq y$ and $y \neq z$, then it holds that $(x \diamond_R^i y)$ and $(y \diamond_R^i z)$. Since \diamond_R is transitive, it holds that $x \diamond_R z$. As $xR^c x$ and $zR^c z$, it follows that $x \diamond_R^i z$. Hence, $x(\diamond_R^i \cup \delta)z$.

We conclude that $\diamond_R^i \cup \delta$ is transitive.

- (iii) Let $x, y, z \in X$ be such that $x(\circ_R^r \cup \diamond_R^i \cup \delta)y$ and $y(\circ_R^r \cup \diamond_R^i \cup \delta)z$.
- (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\circ_R^r \cup \diamond_R^i \cup \delta)z$.
- (b) If $x \neq z$, $x \neq y$ and $y \neq z$, then since $(x \circ_R^r y \wedge y \diamond_R^i z)$ and $(x \diamond_R^i y \wedge y \circ_R^r z)$ are two impossible cases, it follows that $(x(\circ_R^r \cup \delta)y \wedge y(\circ_R^r \cup \delta)z)$ or $(x(\diamond_R^i \cup \delta)y \wedge y(\diamond_R^i \cup \delta)z)$. From (i) and (ii), it follows that $x(\circ_R^r \cup \delta)z$ or $x(\diamond_R^i \cup \delta)z$. Hence, it holds that $x(\circ_R^r \cup \diamond_R^i \cup \delta)z$.

We conclude that $\circ_R^r \cup \diamond_R^i \cup \delta$ is transitive. \square

From Corollary 3.5 and Proposition 3.9, the following result follows.

Corollary 3.10. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then it holds that $\circ_R^r \cup \delta$, $\diamond_R^i \cup \delta$ and $\circ_R^r \cup \diamond_R^i \cup \delta$ are equivalence relations.*

4. Compatibility of a Relation with the Two Auxiliary Relations Associated with Its Clone Relation

In this section, we study the compatibility of a relation with the two subrelations \circ_R^r and \diamond_R^i associated with its clone relation.

4.1. Compatibility of Fuzzy Relations. In this subsection, we recall some basic definitions and results on compatibility, right compatibility and left compatibility of two L -relations on a universe X . Further information can be found in [11, 16]. We pay particular attention to the case where the first L -relation considered is a crisp relation, in particular a pseudo-order relation.

Definition 4.1. [16] Let R_1 and R_2 be two L -relations on a universe X .

- (i) R_1 is called left compatible with R_2 , denoted $R_1 \nabla_l R_2$, if it holds that $R_1(x, y) * R_2(x, z) \leq R_1(z, y)$, for any $x, y, z \in X$.
- (ii) R_1 is called right compatible with R_2 , denoted $R_1 \nabla_r R_2$, if it holds that $R_1(x, y) * R_2(y, t) \leq R_1(x, t)$, for any $x, y, t \in X$.
- (iii) R_1 is called compatible with R_2 , denoted $R_1 \nabla R_2$, if it holds that $R_1(x, y) * R_2(x, z) * R_2(y, t) \leq R_1(z, t)$, for any $x, y, z, t \in X$.

Lemma 4.2. [16] *Let R_1 and R_2 be two L -relations on a universe X . Then it holds that*

- (i) *If $R_1 \nabla_l R_2$ and $R_1 \nabla_r R_2$, then $R_1 \nabla R_2$.*
- (ii) *If $R_1 \nabla R_2$ and R_2 is reflexive, then $R_1 \nabla_l R_2$ and $R_1 \nabla_r R_2$.*

Lemma 4.3. [16] *Let R be an L -relation on a universe X and $(S_i)_{i \in I}$ be a family of L -relations on X . Then it holds that*

- (i) $R \nabla_\tau S_i$, for any $i \in I$, if and only if $R \nabla_\tau \left(\bigcup_{i \in I} S_i \right)$.
- (ii) $R \nabla_l S_i$, for any $i \in I$, if and only if $R \nabla_l \left(\bigcup_{i \in I} S_i \right)$.

Let R be a relation on a set X and E be an L -relation on X , then compatibility of R with E states that

$$\tau(xRy) * E(x, z) * E(y, t) \leq \tau(zRt), \quad (1)$$

for any $x, y, z, t \in X$. Note that throughout this paper, we use the notation τ to refer to the characteristic mapping of a relation R on a set X , i.e., $\tau(xRy) = 1$ if xRy , while $\tau(xRy) = 0$ if $xR^c y$.

The following theorem shows that the crisp equality is the only reflexive L -relation that a pseudo-order relation is compatible with.

Theorem 4.4. *A pseudo-order relation R on a set X is compatible with a reflexive L -relation on X if and only if E is the crisp equality on X .*

Proof. Suppose that R is compatible with E . It follows that

$$\tau(xRx) * E(x, x) * E(x, y) \leq \tau(xRy),$$

for any $x, y \in X$. Since R and E are reflexive, it follows that $E(x, y) \leq \tau(xRy)$, for any $x, y \in X$. Similarly, it holds that $E(x, y) \leq \tau(yRx)$, for any $x, y \in X$. Hence, $E(x, y) \leq \min(\tau(xRy), \tau(yRx))$, for any $x, y \in X$. On the one hand, since R is antisymmetric, it follows that $E(x, y) = 0$, for any $x, y \in X$ such that $x \neq y$. On the other hand, since E is reflexive, it holds that $E(x, x) = 1$, for any $x \in X$. Therefore, E is the crisp equality on X .

Conversely, it is obvious that R is compatible with the crisp equality. \square

The above theorem implies the following corollary shown earlier in [16].

Corollary 4.5. *An order relation R on a set X is compatible with an L -tolerance or L -equivalence relation E if and only if E is the crisp equality on X .*

4.2. Compatibility of a Relation R with the Relations \circ_R^r and \diamond_R^i . In this subsection, we prove some key results, which will be helpful in the proofs of our main theorems in Section 5.

Proposition 4.6. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:*

- (i) R is compatible with \circ_R^r .
- (ii) R is compatible with \diamond_R^i .

Proof. (i) In view of Lemma 4.2, it suffices to prove that $R \nabla_l \circ_R^r$ and $R \nabla_\tau \circ_R^r$.

(a) Let $x, y, z \in X$, then we need to prove that

$$\tau(xRy) * \tau(x \circ_R^r z) \leq \tau(zRy).$$

- (1) If $xR^c y$ or $x(\circ_R^r)^c z$, then $\tau(xRy) * \tau(x \circ_R^r z) = 0$ and the inequality is trivially fulfilled.

- (2) If xRy and $x \circ_R^r z$, then we distinguish three cases: $x \neq y \neq z$, $x = y$ and $y = z$:
- (α) If $x \neq y \neq z$, then since xRy and $x \approx_R z$, it follows that zRy . Hence, it holds that $\tau(xRy) * \tau(x \circ_R^r z) \leq \tau(zRy) = 1$.
 - (β) If $x = y$, then, by definition of \circ_R^r , it follows that $\tau(x \circ_R^r z) \leq \tau(zRx)$. Hence, it holds that $\tau(xRx) * \tau(x \circ_R^r z) \leq \tau(zRx)$.
 - (γ) If $y = z$, then, by definition of \circ_R^r , it follows that $\tau(x \circ_R^r y) \leq \tau(yRy)$. Hence, it holds that $\tau(xRy) * \tau(x \circ_R^r y) \leq \tau(yRy)$.

We conclude that $R \nabla_l \circ_R^r$.

- (b) Let $x, y, t \in X$, then we need to prove that

$$\tau(xRy) * \tau(y \circ_R^r t) \leq \tau(xRt).$$

- (1) If $xR^c y$ or $y(\circ_R^r)^c t$, then $\tau(xRy) * \tau(y \circ_R^r t) = 0$ and the inequality is trivially fulfilled.
- (2) If xRy and $y \circ_R^r t$, then we distinguish the following cases:
 - (α) If $y \neq x \neq t$, then since xRy and $y \approx_R t$, it follows that xRt . Hence, it holds that $\tau(xRy) * \tau(y \circ_R^r t) \leq \tau(xRt) = 1$.
 - (β) If $x = y$, then, by definition of \circ_R^r , it follows that $\tau(x \circ_R^r t) \leq \tau(xRt)$. Hence, it holds that $\tau(xRx) * \tau(x \circ_R^r t) \leq \tau(xRt)$.
 - (γ) If $x = t$, then, by definition of \circ_R^r , it follows that $\tau(y \circ_R^r x) \leq \tau(xRx)$. Hence, it holds that $\tau(xRy) * \tau(y \circ_R^r x) \leq \tau(xRx)$.

We conclude that $R \nabla_l \circ_R^r$.

- (ii) Can be proved analogously. □

Combining Proposition 4.6, Lemma 4.3 and the fact that R is compatible with δ , we obtain the following corollary.

Corollary 4.7. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then it holds that R is compatible with $(\circ_R^r \cup \diamond_R^i \cup \delta)$.*

From Propositions 3.3 and 4.6, we obtain the following corollary.

Corollary 4.8. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then the following statements hold:*

- (i) *If R is reflexive, then R is compatible with $\circ_R \cup \delta$.*
- (ii) *If R is irreflexive, then R is compatible with $\diamond_R \cup \delta$.*

5. Compatibility of a Relation with an L -tolerance or L -equivalence Relation

In this section, we study the compatibility of an arbitrary relation with an L -tolerance or L -equivalence relation.

5.1. Compatibility of a Relation with an L -tolerance Relation. In this subsection, we characterize the L -tolerance relations that a given relation is compatible with.

Theorem 5.1. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an L -tolerance relation on X . Then it holds that R is compatible with E if and only if $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$.*

Proof. Suppose that R is compatible with E , i.e., $\tau(xRy) * E(x, z) * E(y, t) \leq \tau(zRt)$, for any $x, y, z, t \in X$. Let $a, b \in X$. If $a = b$ or $E(a, b) = 0$, then it trivially holds that $E(a, b) \leq \tau(a(\circ_R^r \cup \diamond_R^i \cup \delta)b)$. If $a \neq b$ and $E(a, b) > 0$, then we need to prove that $\tau(a(\circ_R^r \cup \diamond_R^i)b) = 1$, i.e., it holds that $(aRb \wedge bRa \wedge aRa \wedge bRb)$ or $(a \parallel b \wedge aR^c a \wedge bR^c b)$, as well as $a \approx_R b$.

- (i) First, we prove that $(aRb \wedge bRa \wedge aRa \wedge bRb)$ or $(a \parallel b \wedge aR^c a \wedge bR^c b)$.

On the one hand, since E is symmetric and R is compatible with E , it follows that

$$\tau(aRb) * \underbrace{E(a, b)}_{>0} * \underbrace{E(b, a)}_{>0} \leq \tau(bRa).$$

This implies that $\tau(aRb) \leq \tau(bRa)$. On the other hand, it follows that

$$\tau(bRa) * \underbrace{E(b, a)}_{>0} * \underbrace{E(a, b)}_{>0} \leq \tau(aRb),$$

whence also $\tau(bRa) \leq \tau(aRb)$. Therefore, $\tau(aRb) = \tau(bRa)$, i.e., $(aRb \wedge bRa)$ or $a \parallel b$.

- (a) If $(aRb \wedge bRa)$, then we need to prove that $(aRa \wedge bRb)$.

On the one hand, since E is reflexive and R is compatible with E , it follows that

$$\underbrace{\tau(aRb)}_{=1} * \underbrace{E(a, a)}_{=1} * \underbrace{E(b, a)}_{>0} \leq \tau(aRa).$$

This implies that $\tau(aRa) = 1$, i.e., aRa . On the other hand, it follows that

$$\underbrace{\tau(bRa)}_{=1} * \underbrace{E(b, b)}_{=1} * \underbrace{E(a, b)}_{>0} \leq \tau(bRb),$$

whence also bRb .

- (b) If $a \parallel b$, then, in an analogous way, we can prove that $(aR^c a \wedge bR^c b)$.

We conclude that $(aRb \wedge bRa \wedge aRa \wedge bRb)$ or $(a \parallel b \wedge aR^c a \wedge bR^c b)$.

- (ii) Second, it remains to prove that $a \approx_R b$. Let $c \in X \setminus \{a, b\}$. We distinguish four subcases.

- (a) If aRc , then since E is reflexive and R is compatible with E , it follows that

$$0 < E(a, b) = \underbrace{\tau(aRc)}_{=1} * \underbrace{E(a, b)}_{>0} * \underbrace{E(c, c)}_{=1} \leq \tau(bRc).$$

This implies that $\tau(bRc) > 0$, whence bRc .

- (b) If bRc , then since E is reflexive, symmetric and R is compatible with E , it follows that

$$0 < E(a, b) = \underbrace{\tau(bRc)}_{=1} * \underbrace{E(b, a)}_{=E(a,b)>0} * \underbrace{E(c, c)}_{=1} \leq \tau(aRc).$$

This implies that $\tau(aRc) > 0$, whence aRc .

- (c) If cRa , then since E is reflexive and R is compatible with E , it follows that

$$0 < E(a, b) = \underbrace{\tau(cRa)}_{=1} * \underbrace{E(c, c)}_{=1} * \underbrace{E(a, b)}_{>0} \leq \tau(cRb).$$

This implies that $\tau(cRb) > 0$, whence cRb .

- (d) If cRb , then since E is reflexive, symmetric and R is compatible with E , it follows that

$$0 < E(a, b) = \underbrace{\tau(cRb)}_{=1} * \underbrace{E(c, c)}_{=1} * \underbrace{E(b, a)}_{=E(a,b)>0} \leq \tau(cRa).$$

This implies that $\tau(cRa) > 0$, whence cRa .

We conclude that $a \approx_R b$.

Finally, we conclude that $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$.

Conversely, if $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$, then due to the monotonicity of $*$, it holds that

$$\tau(xRy) * E(x, z) * E(y, t) \leq \tau(xRy) * \tau(\circ_R^r \cup \diamond_R^i \cup \delta)z * \tau(y(\circ_R^r \cup \diamond_R^i \cup \delta)t),$$

for any $x, y, z, t \in X$. From Corollary 4.7, it follows that

$$\tau(xRy) * E(x, z) * E(y, t) \leq \tau(zRt),$$

for any $x, y, z, t \in X$. Hence, R is compatible with E . \square

From Corollary 3.5 and Theorem 5.1, the following result is straightforward.

Corollary 5.2. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Then it holds that $\circ_R^r \cup \diamond_R^i \cup \delta$ is the greatest L -tolerance relation on X that R is compatible with.*

Proposition 3.3 and Theorem 5.1 lead to the following corollary.

Corollary 5.3. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an L -tolerance relation on X . Then the following statements hold:*

- (i) *If R is reflexive, then R is compatible with E if and only if $E \subseteq \circ_R \cup \delta$.*
- (ii) *If R is irreflexive, then R is compatible with E if and only if $E \subseteq \diamond_R \cup \delta$.*

Combining Proposition 3.7 and Theorem 5.1, we obtain the following corollary.

Corollary 5.4. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an L -tolerance relation on X . If R is compatible with E , then it holds that the relations R^t , R^c and R^d are compatible with E .*

The following lemma will be useful in the proof of our representation theorem, i.e., the representation of the L -tolerance relations a given relation is compatible with. Its proof is straightforward.

Lemma 5.5. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Let α and β be two L -tolerance relations on X such that $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$, then the union $E = \alpha \cup \beta$ is the L -tolerance relation on X given by*

$$E(x, y) = \begin{cases} 1 & , \text{ if } x = y, \\ \alpha(x, y) & , \text{ if } x \circ_R^r y, \\ \beta(x, y) & , \text{ if } x \diamond_R^i y, \\ 0 & , \text{ otherwise.} \end{cases}$$

Theorem 5.6. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an L -tolerance relation on X . Then it holds that R is compatible with E if and only if there exist two L -tolerance relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Proof. Suppose that R is compatible with E . Consider the L -relations α and β on X defined by

$$\alpha(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \circ_R^r \cup \delta, \\ 0 & , \text{ otherwise,} \end{cases}$$

$$\beta(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \diamond_R^i \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

Note that if $(x, y) \in \circ_R^r \cup \delta$, then it also holds that $(y, x) \in \circ_R^r \cup \delta$. Similarly, if $(x, y) \in \diamond_R^i \cup \delta$, then also $(y, x) \in \diamond_R^i \cup \delta$. Hence, α and β are L -tolerance relations on X . Since R is compatible with E , it follows from Theorem 5.1 that $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$, whence $E(x, y) = 0$ if $x(\circ_R^r \cup \diamond_R^i \cup \delta)^c y$. Hence, it follows that $E = \alpha \cup \beta$.

Conversely, let $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ be two L -tolerance relations on X . Lemma 5.5 implies that $E = \alpha \cup \beta$ is an L -tolerance relation on X . Since $E \subseteq \circ_R^r \cup \diamond_R^i \cup \delta$, Theorem 5.1 guarantees that R is compatible with E . \square

As a corollary, we obtain the following representation of the crisp tolerance relations a given relation is compatible with.

Corollary 5.7. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be a tolerance relation on X . Then it holds that R is compatible with E if and only there exist two L -tolerance relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Remark 5.8. In the setting of Corollary 5.7, we have:

- (i) If $\alpha = \delta$ and $\beta = \delta$, then $E = \delta$.
- (ii) If $\alpha = \circ_R^r \cup \delta$ and $\beta = \diamond_R^i \cup \delta$, then $E = \circ_R^r \cup \diamond_R^i \cup \delta$.

Combining Proposition 3.3 and Theorem 5.6, we obtain the following corollaries.

Corollary 5.9. *Let R be a reflexive relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and E be an L -tolerance relation on X . Then it holds that R is compatible with E if and only if there exists an L -tolerance relation α on X with $\alpha \subseteq \circ_R \cup \delta$ such that*

$$E(x, y) = \begin{cases} \alpha(x, y) & , \text{ if } (x, y) \in \circ_R \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

Corollary 5.10. *Let R be an irreflexive relation on a set X , \approx_R be the clone relation of R with corresponding \diamond_R^i and E be an L -tolerance relation on X . Then it holds that R is compatible with E if and only if there exists an L -tolerance relation β on X with $\beta \subseteq \diamond_R \cup \delta$ such that*

$$E(x, y) = \begin{cases} \beta(x, y) & , \text{ if } (x, y) \in \diamond_R \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

5.2. Compatibility of a Relation with an L -equivalence Relation. In this subsection, we characterize the L -equivalence relations a given relation is compatible with.

Proposition 5.11. *Let R be a relation on a set X and \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i . Let α and β two L -equivalence relations on X such that $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$, then the union $E = \alpha \cup \beta$ is an L -equivalence relation on X .*

Proof. Due to Lemma 5.5, E is an L -tolerance relation. It remains to show that E is $*$ -transitive. Let $x, y, z \in X$, then we need to show that

$$E(x, y) * E(y, z) \leq E(x, z).$$

Let $x, y, z \in X$ such that $E(x, y) > 0$ and $E(y, z) > 0$. We consider the following cases.

- (i) If $x = y$ or $y = z$ or $x = z$, then the inequality $E(x, y) * E(y, z) \leq E(x, z)$ trivially holds.
- (ii) Suppose that $x \neq y$, $y \neq z$ and $x \neq z$. Since $E \subseteq \circ_R^r \cup \diamond_R^i$, it follows that $(x \circ_R^r y \wedge y \circ_R^r z)$ or $(x \circ_R^r y \wedge y \diamond_R^i z)$ or $(x \diamond_R^i y \wedge y \circ_R^r z)$ or $(x \diamond_R^i y \wedge y \diamond_R^i z)$.
 - (a) From the definition of \circ_R^r and \diamond_R^i , it follows that the cases $(x \circ_R^r y \wedge y \diamond_R^i z)$ and $(x \diamond_R^i y \wedge y \circ_R^r z)$ are impossible. Otherwise, it would follow that $(yRy \wedge yR^c y)$, a contradiction.
 - (b) If $(x \circ_R^r y \wedge y \circ_R^r z)$, then it follows that $x \circ_R^r y, y \approx_R z, zRz$ and $z \in X \setminus \{x, y\}$. From Proposition 3.8, it follows that $x \circ_R^r z$. Hence, $E(x, y) = \alpha(x, y)$, $E(y, z) = \alpha(y, z)$ and $E(x, z) = \alpha(x, z)$. Since α is $*$ -transitive, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$, i.e., $E(x, y) * E(y, z) \leq E(x, z)$.

- (c) If $(x \diamond_R^i y \wedge y \diamond_R^i z)$, then it follows that $x \diamond_R^i y$, $y \approx_R z$, $zR^c z$ and $z \in X \setminus \{x, y\}$. From Proposition 3.8, it follows that $x \diamond_R^i z$. Hence, $E(x, y) = \beta(x, y)$, $E(y, z) = \beta(y, z)$ and $E(x, z) = \beta(x, z)$. Since β is $*$ -transitive, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$, i.e., $E(x, y) * E(y, z) \leq E(x, z)$.

We conclude that $E = \alpha \cup \beta$ is an L -equivalence relation on X . \square

Combining Theorem 5.6 and Proposition 5.11 easily leads to the following theorem.

Theorem 5.12. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an L -equivalence relation on X . Then it holds that R is compatible with E if and only if there exist two L -equivalence relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Proof. Theorem 5.6 states that R is compatible with E if and only if there exist two L -tolerance relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$, where

$$\alpha(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \circ_R^r \cup \delta, \\ 0 & , \text{ otherwise,} \end{cases}$$

$$\beta(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in \diamond_R^i \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

Let $x, y, z \in X$, then we need to show that

$$\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$$

and

$$\beta(x, y) * \beta(y, z) \leq \beta(x, z).$$

- (i) First, we prove that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$.
- (a) If $x = y$ or $y = z$ or $x = z$, then this inequality trivially holds.
 - (b) Suppose that $x \neq y$, $y \neq z$ and $x \neq z$.
 - (1) If $x(\circ_R^r)^c y$ or $y(\circ_R^r)^c z$, then it follows that $\alpha(x, y) = 0$ or $\alpha(y, z) = 0$. Hence, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$.
 - (2) If $x \circ_R^r y$ and $y \circ_R^r z$, then it follows that $x \circ_R^r y$, $y \approx_R z$, $zR^c z$ and $z \in X \setminus \{x, y\}$. From Proposition 3.8, it follows that $x \circ_R^r z$. As $x \circ_R^r y$, $y \circ_R^r z$ and $x \circ_R^r z$, it follows that $\alpha(x, y) = E(x, y)$, $\alpha(y, z) = E(y, z)$ and $\alpha(x, z) = E(x, z)$. Since E is $*$ -transitive, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$.
- Thus, α is $*$ -transitive.
- (ii) Second, we prove that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$.
- (a) If $x = y$ or $y = z$ or $x = z$, then this inequality trivially holds.
 - (b) Suppose that $x \neq y$, $y \neq z$ and $x \neq z$.
 - (1) If $x(\diamond_R^i)^c y$ or $y(\diamond_R^i)^c z$, then it follows that $\beta(x, y) = 0$ or $\beta(y, z) = 0$. Hence, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$.

- (2) If $x \diamond_R^i y$ and $y \diamond_R^i z$, then it follows that $x \diamond_R^r y$, $y \approx z$, $zR^c z$ and $z \in X \setminus \{x, y\}$. From Proposition 3.8, it follows that $x \diamond_R^i z$. As $x \diamond_R^i y$, $y \diamond_R^i z$ and $x \diamond_R^i z$, it follows that $\beta(x, y) = E(x, y)$, $\beta(y, z) = E(y, z)$ and $\beta(x, z) = E(x, z)$. Since E is $*$ -transitive, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$.

Thus, β is $*$ -transitive.

We conclude that α and β are L -equivalence relations on X .

For the converse, Proposition 5.11 guarantees that $E = \alpha \cup \beta$ is an L -equivalence relation on X . \square

As a corollary, we obtain the following representation of the L -equality or equivalence relations a relation is compatible with.

Corollary 5.13. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an L -equality relation on X . Then it holds that R is compatible with E if and only if there exist two L -equality relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

Corollary 5.14. *Let R be a relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and \diamond_R^i and E be an equivalence relation on X . Then it holds that R is compatible with E if and only if there exist two equivalence relations α and β on X with $\alpha \subseteq \circ_R^r \cup \delta$ and $\beta \subseteq \diamond_R^i \cup \delta$ such that $E = \alpha \cup \beta$.*

From Proposition 3.3 and Theorem 5.12, we obtain the following representation of the L -equivalence relations that a reflexive or irreflexive relation is compatible with.

Corollary 5.15. *Let R be a reflexive relation on a set X , \approx_R be the clone relation of R with corresponding \circ_R^r and E be an L -equivalence relation on X . Then it holds that R is compatible with E if and only if there exists an L -equivalence relation α on X with $\alpha \subseteq \circ_R^r \cup \delta$ such that*

$$E(x, y) = \begin{cases} \alpha(x, y) & , \text{ if } (x, y) \in \circ_R^r \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

Corollary 5.16. *Let R be an irreflexive relation on a set X , \approx_R be the clone relation of R with corresponding \diamond_R^i and E be an L -equivalence relation on X . Then it holds that R is compatible with E if and only if there exists an L -equivalence relation β on X with $\beta \subseteq \diamond_R^i \cup \delta$ such that*

$$E(x, y) = \begin{cases} \beta(x, y) & , \text{ if } (x, y) \in \diamond_R^i \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

6. Conclusion

In this work, we have tackled and solved the general problem of characterizing the L -tolerance and L -equivalence relations a given relation is compatible with. To that end, we have expanded our knowledge on the clone relation of a relation by

studying two important subrelations of the partition of this clone relation, informally described as reflexive related clones and irreflexive unrelated clones.

Future work will be directed towards the characterization of the L -tolerance and L -equivalence relations a given L -relation is compatible with. This requires a lot of preparatory work, in particular the proper generalization of the notion of clone relation from crisp relations to L -relations.

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