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The clone relation of a binary relation

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ABSTRACT

In a recent paper, De Baets et al. introduced the clone relation of a strict order relation. Two elements of a poset are said to be a pair of clones (or to be clones) if every other element that is greater (resp. smaller) than one of them is also greater (resp. smaller) than the other one. This clone relation played a key role in the characterization of the *L*-fuzzy tolerance relations and the *L*-fuzzy equivalence relations that a strict order relation is compatible with. In this paper, we extend the notion of clone relation to any binary relation. Although the definition of such extension is trivial, the corresponding properties significantly differ from those of the clone relation of a strict order relation. We analyse the most important ones among these properties, paying particular attention to a partition of the clone relation in terms of three different types of pairs of clones.

1. Introduction

The clone relation of a strict order relation was recently introduced by De Baets et al. [5]. This notion is based on how elements are related w.r.t. each other in a partially ordered set (poset, for short). Two elements of a poset are said to form a pair of clones (or to be clones, for short) if every other element that is greater (resp. smaller) than one of them is also greater (resp. smaller) than the other one. The clone relation of a strict order relation always is a tolerance relation and it is built up by two different types of pairs of clones: pairs of comparable clones (which constitute an antitransitive relation) and pairs of incomparable clones (which constitute a transitive relation). This partition of the clone relation played a key role in the characterization of the *L*-fuzzy tolerance relations and the *L*-fuzzy equivalence relations that a strict order relation is compatible with. Extending the definition of the clone relation of a strict order relation to an arbitrary binary relation is a trivial task. Nevertheless, when doing so, its properties significantly vary from these of the clone relation of a strict order relation. For instance, this extension leads to the distinction between two different types of pairs of comparable clones: pairs of clones in which one element is related to the other and not the other way around (which constitute an antitransitive relation) and pairs of clones in which both elements are related to each other (which constitute a transitive relation). The main aim of the present paper is to extend the notion of clone relation of a strict order relation to an arbitrary binary relation and to analyse the properties of such extension. This extension is a prerequisite to the characterization of the *L*-fuzzy tolerance relations and the *L*-fuzzy equivalence relations that a given binary relation is compatible with.

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When restricting to a total order relation, the clone relation coincides with the covering relation, i.e. two elements are clones if and only if they are consecutive. This notion of consecutive elements in a totally ordered set was already independently considered in the field of social choice theory by Tideman under the same name: clones. Clones are important in the field of social choice theory since they can easily change the result of an election. Several methods have been proposed in order to guarantee the independence of clones (see [12,13,15]).

Outside the field of social choice theory, the notions of left and right trace of a binary relation were introduced by Doignon et al. [6] based on a concept similar to that of the clone relation. This notion played a key role in the characterization of the basic properties of a fuzzy relation and of the compatibility of fuzzy relations (see [2,7,9]).

The rest of the paper is structured as follows. After recalling some basic definitions and properties in Section 2, we extend the notion of clone relation of a strict order relation to an arbitrary binary relation in Section 3. In Section 4, we introduce the partition of the clone relation in terms of three different types of pairs of clones. In Section 5, we characterize the clone relation of the three different types of disjoint union. Finally, we present some conclusions and we discuss future research in Section 6.

2. Basic concepts

This section serves an introductory purpose. First, we recall some basic concepts and properties of a binary relation. Second, the notion of clone relation of a strict order relation introduced by De Baets et al. [5] is briefly recalled.

2.1. Binary relations

A binary relation on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements, denoted by $x \nparallel y$, if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \|_R y$, or simply $x \| y$ when no confusion can occur. We denote by R^c the complement of the relation R on X, i.e., for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R on X, i.e., for any $x, y \in X$, xR^ty denotes the fact that yRx. We denote by R^d the dual of the relation R on X, i.e., for any $x, y \in X$, xR^dy denotes the fact that yR^cx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. If $R \cap S = \emptyset$, then R and S are called disjoint relations. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$.

A binary relation R on a set X is called:

(i) reflexive, if, for any $x \in X$, it holds that xRx;
(ii) irreflexive, if, for any $x \in X$, it holds hat xR^cx;
(iii) symmetric, if, for any $x, y \in X$, it holds that xRy implies that yRx;
(iv) antisymmetric, if, for any $x, y \in X$, it holds that xRy and yRx imply that $x = y$;
(v) asymmetric, if, for any $x, y \in X$, it holds that xRy implies that yR^cx;
(vi) transitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xRz;
(vi) antitransitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xR^cz;
(vii) complete, if, for any $x, y \in X$, either xRy or yRx holds.

A binary relation R on a set X is called:

(i) an order relation if it is reflexive, antisymmetric and transitive;
(ii) a total order relation if it is reflexive, antisymmetric, transitive and complete;
(iii) a tolerance relation if it is reflexive and symmetric;
(iv) an equivalence relation if it is reflexive, symmetric and transitive.

A set X equipped with an order relation $\leq$ is called a partially ordered set (poset, for short), denoted by $(X, \leq)$.

For any tolerance/equivalence relation R on a set X, the tolerance/equivalence class of an element $x \in X$ is given by $[x]_R = \{y \in X \mid xRy\}$.

For more details on binary relations, we refer to [1,3,8,11].

2.2. The clone relation of a strict order relation

In this subsection, we recall the notion of clone relation of a strict order relation introduced by De Baets et al. [5]. Two elements of a poset are called clones (or are said to be a pair of clones) if they are related in the same way with every other element in the poset. More formally, the clone relation $\approx$ of a strict order relation $<$ is the binary relation on X defined by

$$x \approx y \quad \text{if} \quad \begin{cases} (\forall z \in X \setminus \{x, y\})(z < x \Leftrightarrow z < y) \\ \text{and} \\ (\forall z \in X \setminus \{x, y\})(x < z \Leftrightarrow y < z)\,. \end{cases}$$

Note that the clone relation $\approx$ of a strict order relation $<$ is a tolerance relation on X. This clone relation can be partitioned[1] as follows:

$$\approx = \lhd \cup \rhd \cup \diamond \cup \delta \,,$$

where $\delta = \{(x,y) \in X^2 \mid x = y\}$ and the binary relations $\lhd$, $\rhd$ and $\diamond$ are pairwise disjoint relations given by:

$$\lhd = \approx \cap \ll \,,$$
$$\rhd = \approx \cap \gg \,,$$
$$\diamond = \approx \cap \parallel \,,$$

where $\ll = \{(a,b) \in X^2 \mid (a < b) \wedge (\nexists c \in X)(a < c < b)\}$ and $\gg = \ll^t$.

Note that, on the one hand, $\lhd$ and $\rhd$ are irreflexive, antisymmetric and antitransitive and it holds that $\lhd = \rhd^t$. On the other hand, $\diamond$ is irreflexive, symmetric and transitive. Hence, the clone relation of a poset can be partitioned in terms of two types of pairs of clones: pairs of comparable clones ($\lhd \cup \rhd$) and pairs of incomparable clones ($\diamond$).

3. The clone relation of a binary relation

In this section, we extend the notion of clone relation to an arbitrary binary relation. The study of the basic properties of this clone relation and its relation with set operations is also addressed.

3.1. Definition

The analysis of ‘likeness’ is a relevant matter of study in mathematics. Equivalence relations, which form a basic concept in mathematics, define a natural notion of ‘likeness’ grouping elements in equivalence classes. When we drop transitivity and allow an element to be ‘alike’ to two elements that are not ‘alike’ to each other, one does no longer talk about equivalence relations but about tolerance relations. Another natural way of defining such ‘likeness’ is based on how elements are related w.r.t. the other elements. In that way, two elements are said to be ‘alike’ (from now on clones) if they are related in the same way w.r.t. every other element.

Definition 1. Let R be a relation on a set X. The clone relation $\approx_R$ of R is the binary relation on X defined by

$$x \approx_R y \quad \text{if} \quad \begin{cases} (\forall z \in X \setminus \{x,y\})(zRx \Leftrightarrow zRy) \\ \text{and} \\ (\forall z \in X \setminus \{x,y\})(xRz \Leftrightarrow yRz)\,. \end{cases} \tag{1}$$

If $x \approx_R y$, then we say that x and y are clones w.r.t. the relation R.

Remark 1. Let R be a relation on a set X. Then the following statements hold:

(i) For any $x, y \in X$, if $x \approx_R y$, then it holds that

$$(\forall z \in X \setminus \{x,y\})(z \parallel x \Leftrightarrow z \parallel y)\,.$$

(ii) For any set X of two elements, it holds that $\approx_R = X^2$.
(iii) For any set X, it holds that $\approx_{X^2} = \approx_\emptyset = X^2$.

The matrix representation of a binary relation R can be used for illustrating the notion of clone relation and for facilitating the identification of clones in the finite case. Let R be a relation on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ ($n \in \mathbb{N}^* = \{1,2,3,\ldots\}$). For any x_i, $x_j \in X$ with $1 \le i, j \le n$, it holds that[2]

$$R_{ij} = \begin{cases} 1, & \text{if } x_i R x_j\,, \\ 0, & \text{if } x_i R^c x_j\,. \end{cases}$$

By definition, it holds that $x_i \approx_R x_j$ if, and only if, for any $k \notin \{i,j\}$, it holds that $R_{ik} = R_{jk}$ and $R_{ki} = R_{kj}$. This means that x_i and x_j are clones if and only if the row and column corresponding to x_i coincide with the row and column corresponding to x_j, with the exception of the four elements contained in the intersection of these two rows with these two columns. This is illustrated in Fig. 1.

Example 1. Let R be the relation on $X = \{a,b,c,d,e,f\}$ defined by the graph in Fig. 2.

[1] Although the term ‘partition’ is used, any of the binary relations $\lhd$, $\rhd$ and $\diamond$ might be empty.

[2] In this paper, a relation R is identified with its characteristic mapping χ_R, i.e. $\chi_R(x,y) = 1$ means xRy and $\chi_R(x,y) = 0$ means xR^cy. In a finite setting, a relation can be conveniently represented as a matrix such that $R_{ij} = \chi_R(x_i, x_j)$.

$$R = \begin{array}{c} \\ x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_1 & \ldots & x_i & & \ldots & & x_j & \ldots & x_n \end{array} \\ \begin{pmatrix} R_{11} & \ldots & R_{1i} & \ldots & R_{1j} & \ldots & R_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{i1} & \ldots & R_{ii} & \ldots & R_{ij} & \ldots & R_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{j1} & \ldots & R_{ji} & \ldots & R_{jj} & \ldots & R_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{n1} & \ldots & R_{ni} & \ldots & R_{nj} & \ldots & R_{nn} \end{pmatrix} \end{array}$$

Fig. 1. Natural interpretation of the clone relation by means of the matrix representation of R.

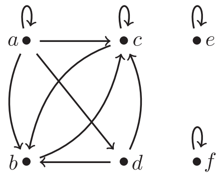

Fig. 2. Graph of a relation R on the set $X = \{a, b, c, d, e, f\}$.

The matrix representation of the relation R is given by:

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Since the row and column corresponding to b coincide with the row and column corresponding to c (without taking the four elements in the intersection of rows and columns into account), it holds that $b \approx_R c$. In general, the clone relation of R is given by:

$$\approx_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

For any relation R, the clone relation of R obviously is reflexive and symmetric. Therefore, the following result is straightforward.

Proposition 1. *Let R be a relation on a set X. The clone relation $\approx_R$ of R is a tolerance relation.*

In general, the clone relation $\approx_R$ does not need to be an equivalence relation, as can be seen in Example 2.

Example 2. Let $X = \{1, 2, 3\}$ and $<$ be the usual strict order relation. We can see that $\approx_<$ is not an equivalence relation. For instance, it holds that $1 \approx_< 2$ and $2 \approx_< 3$, while $1 \not\approx_< 3$.

3.2. Basic properties

In this subsection, we discuss the most relevant properties of the clone relation. First, it is trivial to prove that the clone relation of a relation R always coincides with the clone relation of the complement, the transpose and the dual of R.

Proposition 2. *Let R be a relation on a set X. Then the following statements hold:*

(i) $\approx_{R^c} = \approx_R$.
(ii) $\approx_{R^t} = \approx_R$.
(iii) $\approx_{R^d} = \approx_R$.

Second, it can be proved easily that the reflexivity of R has no impact on the clone relation.

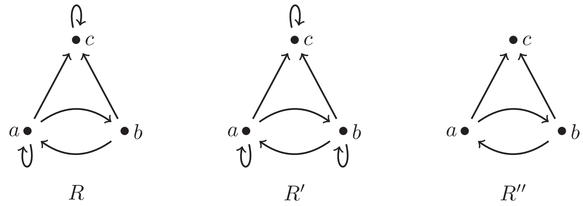

Fig. 3. Graphs of three relations R, R' and R'' on the set $X = \{a, b, c\}$.

Proposition 3. *Let R and S be two relations on a set X. If for any $x, y \in X$ such that $x \neq y$ it holds that $xRy \Leftrightarrow xSy$, then the clone relation of R and the clone relation of S coincide, i.e. $\approx_R = \approx_S$.*

Note that, as a consequence of Proposition 3, we conclude that the clone relation does not take reflexivity or irreflexivity into account. Actually, the relation of an element with itself does not affect the clone relation.

Corollary 1. *Let R, R' and R'' be three relations on a set X. If $R' = R \cup \{(x, x) \in X^2\}$ and $R'' = R \setminus \{(x, x) \in X^2\}$, then it holds that $\approx_R = \approx_{R'} = \approx_{R''}$.*

This result is illustrated in the following example.

Example 3. In Fig. 3, the graphs of three relations R, R' and R'' on the set $X = \{a, b, c\}$ such that R is neither reflexive nor irreflexive, R' is reflexive and R'' is irreflexive are shown. Note that R, R' and R'' coincide for any two different elements. Hence, it holds that

$$\approx_R = \approx_{R'} = \approx_{R''} = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}.$$

Remark 2. If $(X, \leq)$ is a poset and $<$ is the strict order relation associated to the order relation $\leq$, then, from Corollary 1, it follows that $\approx_{\leq} = \approx_{<}$. Note that De Baets et al. [5] defined the clone relation of a poset $(X, \leq)$ by means of the strict order relation $<$ but, in fact, if they had defined this clone relation by means of the order relation $\leq$, then the result would have been the same.

In the following proposition, we study when the clone relation $\approx_R$ is transitive, i.e. when it is an equivalence relation.

Proposition 4. *Let R be a relation on a set X. If there do not exist $x, y \in X$ such that $x \approx_R y$, xRy and $yR^c x$, then it holds that $\approx_R$ is an equivalence relation.*

Proof. Since $\approx_R$ is a tolerance relation (see Proposition 1), it suffices to prove that $\approx_R$ is transitive. Let $x, y, z \in X$ be such that $x \approx_R y$ and $y \approx_R z$. Suppose that $x \not\approx_R z$. It follows that there exists $t \in X \backslash \{x, z\}$ such that (tRx and $tR^c z$) or (xRt and $zR^c t$) or (tRz and $tR^c x$) or (zRt and $xR^c t$).

(i) Let us consider the case where tRx and $tR^c z$. We distinguish two cases: $t \neq y$ and $t = y$.
  (a) If $t \neq y$, then from $x \approx_R y$ and $y \approx_R z$, it follows that tRy and $tR^c y$, a contradiction.
  (b) If $t = y$, then it follows that yRx and $yR^c z$. Since $x \approx_R y$, $y \approx_R z$ and $x \not\approx_R z$, it follows that $x \neq y \neq z \neq x$. Moreover, as $y \approx_R z$, it follows that zRx and $yR^c z$ and, as $x \approx_R y$, this implies that zRy and $yR^c z$. At the same time it holds that $y \approx_R z$, a contradiction with the hypothesis. Therefore, $\approx_R$ is transitive.

(ii) The other cases where (xRt and $zR^c t$) or (tRz and $tR^c x$) or (zRt and $xR^c t$) are analogously proved. □

In particular, the conditions of Proposition 4 are satisfied for any symmetric relation.

Corollary 2. *Let R be a relation on a set X. If R is symmetric, then it holds that $\approx_R$ is an equivalence relation.*

Corollary 3. *Let R be a relation on a set X. If $R = \approx_R$, then it holds that R is an equivalence relation.*

An equivalence relation is always included in its clone relation, as is expressed in the following proposition.

Proposition 5. *Let R be a relation on a set X. If R is an equivalence relation, then it holds that $R \subseteq \approx_R$.*

Proof. Let R be an equivalence relation and $x, y \in X$ be such that xRy. Let us suppose that $x \not\approx_R y$. Since R is an equivalence relation and $x \not\approx_R y$, it follows that there exists $z \in X \backslash \{x, y\}$ such that (zRx and $zR^c y$) or (zRy and $zR^c x$). Due to the symmetry and transitivity of R, it follows that (zRy and $zR^c y$) or (zRx and $zR^c x$), which leads to a contradiction. Hence, it holds that $x \approx_R y$ and, therefore, $R \subseteq \approx_R$. □

The necessary and sufficient conditions that an equivalence relation needs to satisfy in order to coincide with its clone relation are provided in the following proposition. In words, an equivalence relation coincides with its clone relation if and only if there is at most one singleton equivalence class.

Proposition 6. *Let R be a relation on a set X. If R is an equivalence relation, then it holds that $R = \approx_R$ if and only if there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$.*

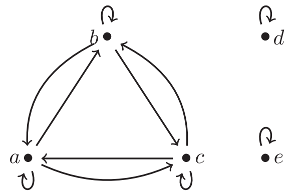

Fig. 4. Graph of an equivalence relation R on the set $X = \{a, b, c, d, e\}$.

Proof. (⇒) Let R be an equivalence relation on X such that $R = \approx_R$ and suppose that there exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$. It follows that xR^cz and that yR^cz for any $z \in X \backslash \{x, y\}$, therefore it holds that $x \approx_R y$, a contradiction with $R = \approx_R$ and xR^cy. Hence, there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = y$.

(⇐) From Proposition 5, it follows that $R \subseteq \approx_R$. It remains to prove that $\approx_R \subseteq R$. Let R be an equivalence relation on X such that there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$. Let us suppose that $\approx_R \not\subseteq R$. As R is reflexive, it holds that there exist $x, y \in X$ such that $x \neq y$, $x \approx_R y$ and xR^cy. Since ($[x]_R \neq \{x\}$ or $[y]_R \neq \{y\}$) and $x \neq y$, it follows that there exists $z \in X \backslash \{x, y\}$ such that xRz or yRz. As $x \approx_R y$, it implies that (xRz and yRz) or (yRz and xRz). Since R is an equivalence relation, it follows that xRy, a contradiction. Hence, $\approx_R \subseteq R$ and, therefore, $\approx_R = R$. □

In general, the fact that R is an equivalence relation does not necessarily lead to $\approx_R \subseteq R$, as can be seen from Example 4.

Example 4. The relation R defined in Fig. 4 is an equivalence relation on the set $X = \{a, b, c, d, e\}$.

The matrix representations of R and $\approx_R$ are given by:

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad \approx_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

We can see that $d \approx_R e$ and dR^ce. Hence, it holds that $\approx_R \not\subseteq R$. Note that it holds that $\approx_R \not\subseteq R$, due to the fact that there are two equivalence classes formed by singletons. Note that, as expected due to Proposition 5, it holds that $R \subseteq \approx_R$.

The composition of any symmetric relation with its clone relation is always included in that relation. In addition, we will prove that the clone relation of any symmetric relation is the greatest symmetric relation that satisfies this inclusion.

Proposition 7. *Let R be a relation on a set X. If R is symmetric, then the following two statements hold:*

(i) *$\approx_R$ is the greatest symmetric relation S such that $R \circ S \subseteq R$.*
(ii) *$\approx_R$ is the greatest symmetric relation S such that $S \circ R \subseteq R$.*

Proof. Let R be a symmetric relation on X.

(i) Let us suppose that there exists a symmetric relation S on X such that $R \circ S \subseteq R$ and $S \not\subseteq \approx_R$. It follows that there exist $x, y \in X$ such that xSy and $x \not\approx_R y$. As $x \not\approx_R y$ and R is symmetric, it follows that there exists $z \in X \backslash \{x, y\}$ such that (zRx and zR^cy) or (zRy and zR^cx). Let us consider, w.l.o.g, that zRx and zR^cy. Since zRx and xSy, it follows that $z(R \circ S)y$. As $R \circ S \subseteq R$, it follows that zRy, a contradiction. Hence, we conclude that $S \subseteq \approx_R$.
(ii) As R and S are symmetric, it holds that $R \circ S = S \circ R$. Therefore, the result follows from statement (i). □

3.3. Interaction of the clone relation with set operations

This subsection is devoted to discuss the interaction of the clone relation with the most common set operations.

Proposition 8. *Let R and S be two relations on a set X. If $R \subseteq S$, then the following statements hold:*

(i) *$\approx_R \cap \approx_{S \backslash R} \subseteq \approx_S$.*
(ii) *$\approx_S \subseteq (\approx_R \cap \approx_{S \backslash R}) \cup ((\approx_R)^c \cap (\approx_{S \backslash R})^c)$.*

Proof.

(i) Suppose that $R \subseteq S$ and let $x, y \in X$ be such that $x(\approx_R \cap \approx_{S \backslash R})y$. It follows that $x \approx_R y$ and $x \approx_{S \backslash R} y$. Therefore, for any $z \in X \backslash \{x, y\}$, it holds that

$$\begin{aligned} xSz &\Leftrightarrow (xRz \vee x(S \setminus R)z) \\ &\Leftrightarrow (yRz \vee y(S \setminus R)z) \\ &\Leftrightarrow ySz\,. \end{aligned}$$

In a similar way, we prove that $zSx \Leftrightarrow zSy$. Hence, it holds that $x \approx_S y$. Therefore, it holds that $\approx_R \cap \approx_{S \backslash R} \subseteq \approx_S$.

(ii) Suppose that $R \subseteq S$ and let $x, y \in X$ be such that $x \approx_S y$. Since it trivially holds that $X^2 = (\approx_R \cup (\approx_R)^c) \cap (\approx_{S\setminus R} \cup (\approx_{S\setminus R})^c)$, it follows that one of the following statements holds: $x(\approx_R \cap \approx_{S\setminus R})y$ or $x(\approx_R \cap (\approx_{S\setminus R})^c)y$ or $x((\approx_R)^c \cap \approx_{S\setminus R})y$ or $x((\approx_R)^c \cap (\approx_{S\setminus R})^c)y$. We will prove that $x((\approx_R)^c \cap \approx_{S\setminus R})^c y$ and $x(\approx_R \cap (\approx_{S\setminus R})^c)^c y$.

(a) Suppose that $(x(\approx_R)^c y$ and $x \approx_{S\setminus R} y)$. Since $x(\approx_R)^c y$, it follows that there exists $z \in X\setminus\{x, y\}$ such that one of the following statements holds: (xRz and yR^cz) or (yRz and xR^cz) or (zRx and zR^cy) or (zRy and zR^cx). Any of these cases contradicts the fact that ($x \approx_S y$ and $x \approx_{S\setminus R} y$). For instance, if ($xRz$ and yR^cz), then, since $R \subseteq S$, it follows that xSz. Since $x \approx_S y$ and $z \in X\setminus\{x, y\}$, it follows that ySz. On the one hand, since ySz and yR^cz, it follows that $y(S\setminus R)z$. On the other hand, since xRz, it follows that $x(S\setminus R)^cz$. a contradiction with the fact that $x \approx_{S\setminus R} y$. The other cases where (yRz and xR^cz) or (zRx and zR^cy) or (zRy and zR^cx) are analogously proved.

(b) Suppose that ($x \approx_R y$ and $x(\approx_{S\setminus R} y)^c$). Since $x(\approx_{S\setminus R})^c y$, it follows that there exists $z \in X\setminus\{x, y\}$ such that one of the following statements holds: ($x(S\setminus R)z$ and $y(S\setminus R)^cz$) or ($y(S\setminus R)z$ and $x(S\setminus R)^cz$) or ($z(S\setminus R)x$ and $z(S\setminus R)^cy$) or ($z(S\setminus R)y$ and $z(S\setminus R)^cx$). Any of these cases contradicts the fact that ($x \approx_S y$ and $x \approx_R y$). For instance, if ($x(S\setminus R)z$ and $y(S\setminus R)^cz$), then it follows that (xSz and xR^cz) and (yS^cz or yRz). Therefore, it holds that (xSz and yS^cz) or (xR^cz and yRz), a contradiction with the fact that ($x \approx_S y$ and $x \approx_R y$). The other cases where ($y(S\setminus R)z$ and $x(S\setminus R)^cz$) or ($z(S\setminus R)x$ and $z(S\setminus R)^cy$) or ($z(S\setminus R)y$ and $z(S\setminus R)^cx$) are analogously proved.

Hence, it holds that ($x \approx_R y$ and $x \approx_{S\setminus R} y$) or ($x(\approx_R)^c y$ and $x(\approx_{S\setminus R})^c y$). Therefore, it holds that $\approx_S \subseteq (\approx_R \cap \approx_{S\setminus R}) \cup ((\approx_R)^c \cap (\approx_{S\setminus R})^c)$. $\square$

The following corollary follows immediately from statement (ii) of Proposition 8.

Corollary 4. *Let R and S be two relations on a set X. Then it holds that*

$$\approx_{R\cup S} \subseteq (\approx_R \cap \approx_{S\setminus R}) \cup ((\approx_R)^c \cap (\approx_{S\setminus R})^c)\,.$$

Note that, in general, if R and S are two binary relations on a set X such that $S \subseteq R$ and $\approx_R$ and $\approx_S$ are their respective clone relations, then it does not necessarily hold that $\approx_S \subseteq \approx_R$, as can be seen in Example 5.

Example 5. Let us consider two binary relations R and S on the set $X = \{a, b, c, d\}$ defined by $R = \{(a, a), (b, b), (a, b), (a, c)\}$ and $S = \{(a, a), (b, b)\}$. It holds that $S \subseteq R$, $a \approx_S b$, while $a(\approx_R)^c b$. Hence, $\approx_S \not\subseteq \approx_R$.

The following corollary follows immediately from Proposition 2.

Corollary 5. *Let $(R_i)_{i\in I}$ be a finite family of relations on a set X. The following statements hold:*

(i) $\approx_{\bigcup\limits_{i\in I} R_i} = \approx_{\bigcap\limits_{i\in I} R_i^c}$.

(ii) $\approx_{\bigcap\limits_{i\in I} R_i} = \approx_{\bigcup\limits_{i\in I} R_i^c}$.

In the following, we discuss the interaction of the clone relation with the intersection and the union.

Proposition 9. *Let R and S be two relations on a set X. The following statements hold:*

(i) $\approx_R \cap \approx_S = \approx_{R\cap S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R}$.

(ii) $\approx_R \cap \approx_S = \approx_{R\cup S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R}$.

Proof.

(i) We need to prove that $\approx_R \cap \approx_S \subseteq \approx_{R\cap S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R}$ and that $\approx_{R\cap S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R} \subseteq \approx_R \cap \approx_S$.

(a) First, we prove that $\approx_R \cap \approx_S \subseteq \approx_{R\cap S} \cap \approx_{R\setminus S} \cap \approx_{S\setminus R}$. Let $x, y \in X$ be such that $x(\approx_R \cap \approx_S)y$. It follows that $x \approx_R y$ and $x \approx_S y$. Therefore, for any $z \in X\setminus\{x, y\}$, it holds that

$$\begin{aligned} x(R \cap S)z &\Leftrightarrow xRz \wedge xSz \\ &\Leftrightarrow yRz \wedge ySz \\ &\Leftrightarrow y(R \cap S)z\,. \end{aligned}$$

In a similar way, we prove that $z(R\cap S)x \Leftrightarrow z(R\cap S)y$. Hence, it holds that $x(\approx_{R\cap S})y$ and, thus, that $\approx_R \cap \approx_S \subseteq \approx_{R\cap S}$. Moreover, for any $z \in X\setminus\{x, y\}$, it holds that

$$\begin{aligned} x(R \setminus S)z &\Leftrightarrow xRz \wedge xS^cz \\ &\Leftrightarrow yRz \wedge yS^cz \\ &\Leftrightarrow y(R \setminus S)z\,. \end{aligned}$$

In a similar way, we prove that $z(R\setminus S)x \Leftrightarrow z(R\setminus S)y$.
Hence, it holds that $x(\approx_{R\setminus S})y$ and, thus, that $\approx_R \cap \approx_S \subseteq \approx_{R\setminus S}$. The fact that $\approx_R \cap \approx_S \subseteq \approx_{S\setminus R}$ is proved in an analogous way.

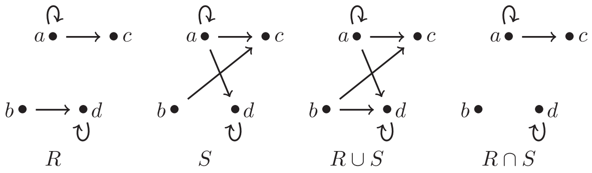

Fig. 5. Graphs of four relations on the set $X = \{a, b, c, d\}$.

(b) Second, we prove that $\approx_{R\cap S}\cap \approx_{R\setminus S}\cap \approx_{S\setminus R} \subseteq \approx_R \cap \approx_S$. We have that

$$\begin{aligned}\approx_{R\cap S}\cap\approx_{R\setminus S}\cap\approx_{S\setminus R} &= \approx_{R\cap S}\cap\approx_{R\setminus(R\cap S)}\cap\approx_{S\setminus(R\cap S)}\\ &= (\approx_{R\cap S}\cap\approx_{R\setminus(R\cap S)})\cap(\approx_{R\cap S}\cap\approx_{S\setminus(R\cap S)}).\end{aligned}$$

From Proposition 8, it follows that $(\approx_{R\cap S}\cap \approx_{R\setminus(R\cap S)}) \subseteq \approx_R$ and $(\approx_{R\cap S}\cap \approx_{S\setminus(R\cap S)}) \subseteq \approx_S$.

(ii) From (i), it follows that $\approx_{R^c}\cap\approx_{S^c}=\approx_{R^c\cap S^c}\cap\approx_{R^c\setminus S^c}\cap\approx_{S^c\setminus R^c}$.
Since $\approx_{R^c}=\approx_R$, $\approx_{S^c}=\approx_S$, $\approx_{R^c\cap S^c}=\approx_{(R\cup S)^c}=\approx_{R\cup S}$, $R^c\setminus S^c = S\setminus R$ and $S^c\setminus R^c = R\setminus S$, it follows that $\approx_R\cap\approx_S=\approx_{R\cup S}\cap\approx_{S\setminus R}\cap\approx_{R\setminus S}$. □

The following corollary is a direct result of Proposition 9.

Corollary 6. *Let R and S be two relations on a set X. The following statements hold:*

(i) $\approx_R\cap \approx_S \subseteq \approx_{R\cap S}$.
(ii) $\approx_R\cap \approx_S \subseteq \approx_{R\cup S}$.

Note that the converse inclusions do not necessarily hold, as can be seen in Example 6.

Example 6. Let R, S, $R\cup S$ and $R\cap S$ be the four relations defined on the set $X = \{a, b, c, d\}$ by the graphs in Fig. 5. We can see that:

(a) $a \approx_{R\cap S} c$, while $a(\approx_S)^c c$. Hence, $\approx_{R\cap S}\not\subseteq\approx_R\cap\approx_S$.
(b) $c \approx_{R\cup S} d$, while $c(\approx_S)^c d$. Hence, $\approx_{R\cup S}\not\subseteq\approx_R\cap\approx_S$.
(c) $a \approx_{R\cup S} b$, while $a(\approx_R)^c b$ and $a(\approx_S)^c b$. Hence, $\approx_{R\cup S}\not\subseteq \approx_R\cup\approx_S$.
(d) $a \approx_R c$, while $a(\approx_{R\cup S})^c c$. Hence, $\approx_R\cup\approx_S\not\subseteq\approx_{R\cup S}$.

4. A partition of the clone relation

De Baets et al. [5] provided a partition of the clone relation for the special case of an order relation. Here, we extend this partition[3] to the case of an arbitrary binary relation.

Definition 2. Let R be a relation on a set X. The following binary relations on X are defined:

(i) $\lhd_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge xRy \wedge yR^c x \wedge x \neq y\}$.
(ii) $\rhd_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge yRx \wedge xR^c y \wedge x \neq y\}$.
(iii) $\circ_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge xRy \wedge yRx \wedge x \neq y\}$.
(iv) $\diamond_R = \{(x, y) \in X^2 \mid x \approx_R y \wedge xR^c y \wedge yR^c x \wedge x \neq y\}$.

Remark 3. Note that $\lhd_R^t = \rhd_R$, $\circ_R^t = \circ_R$ and $\diamond_R^t = \diamond_R$.

Given Definition 2, it is immediately clear that the clone relation $\approx_R$ of any relation R can be written as follows:

$$\approx_R = \lhd_R\cup\rhd_R\cup\circ_R\cup\diamond_R\cup\delta\,,$$

where $\delta = \{(x, y) \in X^2 \mid x = y\}$.

Definition 3. Let R be a relation on a set X. The triplet $(\lhd_R, \circ_R, \diamond_R)$ is called the (canonical) partition of the clone relation $\approx_R$.

Note that in the canonical partition we do not explicitly mention $\rhd_R$ (as it equals $\lhd_R^t$) and δ (as it does not depend on the relation R).

Remark 4. As discussed by Roubens and Vincke [10], any reflexive binary relation Q on a set X allows to partition X^2 into four disjoint parts: a strict preference relation $P_Q = Q\cap(Q^t)^c$ (which is irreflexive and asymmetric) and its transpose P_Q^t, an

[3] Although the term 'partition' is used, any of the binary relations $\lhd_R$, $\rhd_R$, $\circ_R$ and $\diamond_R$ might be empty.

indifference relation $I_Q = Q \cap Q^t$ (which is reflexive and symmetric) and an incomparability relation $J_Q = Q^c \cap (Q^t)^c$ (which is irreflexive and symmetric)

$$X^2 = P_Q \cup P_Q^t \cup I_Q \cup J_Q \,.$$

We can see that the partition of the clone relation is closely related with this result. Indeed, extending the above definition to an arbitrary binary relation R, we can write

$$X^2 = P_R \cup P_R^t \cup (I_R \backslash \delta) \cup J_R \cup \delta \,,$$

and hence

$$\begin{aligned} \approx_R &= \approx_R \cap X^2 \\ &= (\approx_R \cap P_R) \cup (\approx_R \cap P_R^t) \cup (\approx_R \cap (I_R \backslash \delta)) \cup (\approx_R \cap J_R) \cup (\approx_R \cap \delta) \\ &= \lhd_R \cup \rhd_R \cup \circ_R \cup \diamond_R \cup \delta \,. \end{aligned}$$

Example 7. Let R be the relation defined in Example 1. The matrix representations of the relations $\lhd_R$, $\rhd_R$, $\circ_R$ and $\diamond_R$ are given by:

$$\lhd_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \rhd_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\circ_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \diamond_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \,.$$

Note that $\approx_R$ can be written as:

$$\approx_R = \lhd_R + \rhd_R + \circ_R + \diamond_R + \delta \,.$$

From the definition of the partition of the clone relation and from Proposition 2 and Corollary 1, the following results are straightforward

Corollary 7. *Let R be a relation on a set X. Then the following statements hold:*

(i) $(\lhd_{R^c}, \circ_{R^c}, \diamond_{R^c}) = (\rhd_R, \diamond_R, \circ_R)$.
(ii) $(\lhd_{R^t}, \circ_{R^t}, \diamond_{R^t}) = (\rhd_R, \circ_R, \diamond_R)$.
(iii) $(\lhd_{R^d}, \circ_{R^d}, \diamond_{R^d}) = (\lhd_R, \diamond_R, \circ_R)$.

Corollary 8. *Let R, R' and R'' be three relations on a set X. If $R' = R \cup \{(x, x) \in X^2\}$ and $R'' = R \setminus \{(x, x) \in X^2\}$, then it holds that*

(i) $\lhd_R = \lhd_{R'} = \lhd_{R''}$.
(ii) $\rhd_R = \rhd_{R'} = \rhd_{R''}$.
(iii) $\circ_R = \circ_{R'} = \circ_{R''}$.
(iv) $\diamond_R = \diamond_{R'} = \diamond_{R''}$.

Note that, depending on the properties of R, some of the relations $\lhd_R$, $\circ_R$ and $\diamond_R$ may be already determined.

Proposition 10. *Let R be a relation on a set X. The following statements hold:*

(i) *If R is symmetric, then $\lhd_R = \rhd_R = \emptyset$.*
(ii) *If R is antisymmetric, then $\circ_R = \emptyset$.*
(iii) *If R is complete, then $\diamond_R = \emptyset$.*

Proof. Let R be a relation on X.

(i) If R is symmetric, then, for any $x, y \in X$, it holds that xRy and yRx. Hence, it holds that $\lhd_R = \rhd_R = \emptyset$.
(ii) If R is antisymmetric, then, for any $x, y \in X$, it holds that xRy and yRx implies that $x = y$. Hence, it holds that $\circ_R = \emptyset$.
(iii) If R is complete, then, for any $x, y \in X$, it holds that xRy or yRx. Hence, it holds that $\diamond_R = \emptyset$. $\square$

Remark 5. Note that $\circ_R$ was not considered in [5] because an order relation is always antisymmetric. In case the relation R is a total order relation (or, in general, antisymmetric and complete), the relations $\circ_R$ and $\diamond_R$ are no longer relevant as they are empty. In this case, the clone relation coincides with the usual covering relation for (total) order relations, as discussed in [5].

The previous proposition serves to characterize the properties of the clone relation of particular types of binary relations, such as order relations or equivalence relations, in terms of the properties of its partition. For this purpose, we analyse some basic properties of the relations $\lhd_R$, $\rhd_R$, $\circ_R$ and $\diamond_R$.

Theorem 1. *Let R be a relation on a set X. The following statements hold:*

(i) *If $x \lhd_R y$, then, for any $z \in X \backslash \{x, y\}$, $x \approx_R z$ implies that $x \rhd_R z$.*
(ii) *If $x \rhd_R y$, then, for any $z \in X \backslash \{x, y\}$, $x \approx_R z$ implies that $x \lhd_R z$.*
(iii) *If $x \circ_R y$, then, for any $z \in X \backslash \{x, y\}$, $x \approx_R z$ implies that $x \circ_R z$.*
(iv) *If $x \diamond_R y$, then, for any $z \in X \backslash \{x, y\}$, $x \approx_R z$ implies that $x \diamond_R z$.*

Proof.

(i) Let $x, y \in X$ and $z \in X \backslash \{x, y\}$ be such that $x \lhd_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since xRy, $yR^c x$, $x \approx_R z$ and $y \in X \backslash \{x, z\}$, it follows that zRy and $yR^c z$. On the other hand, since zRy, $yR^c z$, $x \approx_R y$ and $z \in X \backslash \{x, y\}$, it follows that zRx and $xR^c z$. As $x \approx_R z$, it follows that $x \rhd_R z$.
(ii) The proof is analogous to that of (i).
(iii) Let $x, y \in X$ and $z \in X \backslash \{x, y\}$ be such that $x \circ_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since xRy, yRx, $x \approx_R z$ and $y \in X \backslash \{x, z\}$, it follows that zRy and yRz. On the other hand, since zRy, yRz, $x \approx_R y$ and $z \in X \backslash \{x, y\}$, it follows that zRx and xRz. As $x \approx_R z$, it follows that $x \circ_R z$.
(iv) Let $x, y \in X$ and $z \in X \backslash \{x, y\}$ be such that $x \diamond_R y$ and $x \approx_R z$. Note that $x \neq y$. On the one hand, since $x \| y$, $x \approx_R z$ and $y \in X \backslash \{x, z\}$, it follows that $z \| y$. On the other hand, since $z \| y$, $x \approx_R y$ and $z \in X \backslash \{x, y\}$, it follows that $z \| x$. As $x \approx_R z$, it follows that $x \diamond_R z$. □

Corollary 9. *Let R be a relation on a set X. Then there are no $x, y, z \in X$ such that $x \lhd_R y$ and $y \lhd_R z$ and $z \lhd_R x$.*

Proof. Suppose that there exist $x, y, z \in X$ such that $x \lhd_R y$, $y \lhd_R z$ and $z \lhd_R x$. Since xRy, $z \approx_R x$ and $y \in X \backslash \{x, z\}$, it follows that zRy, which contradicts $zR^c y$. □

The (ir)reflexivity and (anti)symmetry of the relations $\lhd_R$, $\rhd_R$, $\lhd_R \cup \rhd_R$, $\circ_R$ and $\diamond_R$ is discussed in the following proposition.

Proposition 11. *Let R be a relation on a set X. The following statements hold:*

(i) *$\lhd_R$ is irreflexive and antisymmetric.*
(ii) *$\rhd_R$ is irreflexive and antisymmetric.*
(iii) *$\lhd_R \cup \rhd_R$ is irreflexive and symmetric.*
(iv) *$\circ_R$ is irreflexive and symmetric.*
(v) *$\diamond_R$ is irreflexive and symmetric.*

Proof. By definition, the relations $\lhd_R$, $\rhd_R$, $\diamond_R$, $\circ_R$ and $\lhd_R \cup \rhd_R$ are irreflexive. Next, for any $x, y \in X$, it is immediate to see that both ($x \lhd_R y$ and $y \lhd_R x$) and ($x \rhd_R y$ and $y \rhd_R x$) are impossible; this implies that $\lhd_R$ and $\rhd_R$ are antisymmetric. Since $\lhd_R^t = \rhd_R$ and $\rhd_R^t = \lhd_R$, it follows that $\lhd_R \cup \rhd_R$ is symmetric. In addition, as $\circ_R^t = \circ_R$ and $\diamond_R^t = \diamond_R$, it follows that $\circ_R$ and $\diamond_R$ are symmetric. □

In the following proposition, we discuss the (anti)transitivity of the relations $\lhd_R$, $\rhd_R$, $\lhd_R \cup \rhd_R$, $\circ_R$ and $\diamond_R$.

Proposition 12. *Let R be a relation on a set X. The following statements hold:*

(i) *$\lhd_R$ is antitransitive.*
(ii) *$\rhd_R$ is antitransitive.*
(iii) *$\lhd_R \cup \rhd_R$ is antitransitive.*
(iv) *$\circ_R \cup \delta$ is transitive.*
(v) *$\diamond_R \cup \delta$ is transitive.*

Proof.

(i) Let $x, y, z \in X$ be such that $x \lhd_R y$ and $y \lhd_R z$. Suppose that $x \lhd_R z$. It follows that $x \lhd_R y$, $x \approx_R z$ and $z \in X \backslash \{x, y\}$. Therefore, from Theorem 1, it follows that $x \rhd_R z$, a contradiction. Hence, $\lhd_R$ is antitransitive.
(ii) The proof is analogous to that of (i).
(iii) Let $x, y, z \in X$ be such that $x(\lhd_R \cup \rhd_R)y$ and $y(\lhd_R \cup \rhd_R)z$. From (i) and (ii) it follows that ($x \lhd_R y$ and $y \lhd_R z$) and ($x \rhd_R y$ and $y \rhd_R z$) lead to, respectively, $x(\lhd_R)^c z$ and $x(\rhd_R)^c z$. In addition, due to Corollary 9, we have that ($x \lhd_R y$ and $y \lhd_R z$ and $x \rhd_R z$) and ($x \rhd_R y$ and $y \rhd_R z$ and $x \lhd_R z$) are not possible. On the other hand, if $x \neq z$ then the cases ($x \lhd_R y$ and $y \rhd_R z$) or ($x \rhd_R y$ and $y \lhd_R z$) are not possible, due to Theorem 1; if $x = z$, then $x(\lhd_R \cup \rhd_R)^c x$, due to the irreflexivity of $\lhd_R \cup \rhd_R$. We conclude that $\lhd_R \cup \rhd_R$ is antitransitive.

(iv) Let $x, y, z \in X$ be such that $x(\circ_R \cup \delta)y$ and $y(\circ_R \cup \delta)z$.
  (a) If $x = z$ or $x = y$ or $y = z$, then it trivially holds that $x(\circ_R \cup \delta)z$.
  (b) The case $x \neq z$, $x \neq y$ and $y \neq z$. First, we prove that $x \approx_R z$. Suppose that $x \not\approx_R z$, then it follows that there exists $t \in X \backslash \{x, z\}$ such that (tRx and $tR^c z$) or (tRz and $tR^c x$) or (xRt and $zR^c t$) or (zRt and $xR^c t$). If, for instance, (tRx and $tR^c z$), then, since yRz, it follows that $t \neq y$. As tRx, $tR^c z$, $x \approx_R y$, $y \approx_R z$ and $t \in X \backslash \{x, y, z\}$, it follows that tRy and $tR^c y$, a contradiction. The other cases where (tRz and $tR^c x$) or (xRt and $zR^c t$) or (zRt and $xR^c t$) are analogously proved. We conclude that $x \approx_R z$. Second, as $z \circ_R y$, $z \approx_R x$ and $x \in X \backslash \{y, z\}$, it follows from Theorem 1 that $x \circ_R z$.

  We conclude that $x(\circ_R \cup \delta)z$ and, therefore, $\circ_R \cup \delta$ is transitive.

(v) The proof is similar to that of (iv). □

From Propositions 11 and 12, the following result follows.

Corollary 10. *Let R be a relation on a set X. Then it holds that $(\lhd_R \cup \rhd_R \cup \delta)$ is a tolerance relation and that $(\circ_R \cup \delta)$ and $(\diamond_R \cup \delta)$ are equivalence relations.*

For any $x, y \in X$, there exists at most one element z such that $x \approx_R z$ and $z \approx_R y$ and $x \not\approx_R y$.

Proposition 13. *Let R be a relation on a set X. For any two elements $x, y \in X$, if there exists $z \in X$ such that $x \approx_R z$, $z \approx_R y$ and $x \not\approx_R y$, then it holds that ($x \lhd_R z$ and $z \lhd_R y$) or that ($y \lhd_R z$ and $z \lhd_R x$), that z is unique and that $[z]_{\approx_R} = \{x, y, z\}$.*

Proof. Let $x, y \in X$ be such that there exists $z \in X$ such that $x \approx_R z$, $z \approx_R y$ and $x \not\approx_R y$. This implies that $x \neq z \neq y \neq x$. Hence, ($x \lhd_R z$ or $x \rhd_R z$ or $x \circ_R z$ or $x \diamond_R z$) and $z \approx_R y$. From Theorem 1 and Corollary 10, it follows that ($x \circ_R z$ or $x \diamond_R z$) and $z \approx_R y$ implies $y \approx_R x$, a contradiction. We only need to consider the cases ($x \lhd_R z$ or $x \rhd_R z$) and $z \approx_R y$. From Theorem 1, it follows that ($x \lhd_R z$ and $z \lhd_R y$) or ($y \lhd_R z$ and $z \lhd_R x$) are the only possible cases.

Suppose now that z is not unique, i.e. $\exists z' \in X \backslash \{x, y, z\}$ such that $x \approx_R z'$ and $y \approx_R z'$. Therefore, it holds that ($x \lhd_R z'$ and $z' \lhd_R y$) or ($y \lhd_R z'$ and $z' \lhd_R x$). From Theorem 1, as $x \lhd_R z$ and $x \approx_R z'$, it follows that $x \rhd_R z'$, a contradiction. Hence, $[z]_{\approx_R} = \{x, y, z\}$. □

Next, we provide an important result w.r.t. the structure of the intersection of two tolerance classes of the clone relation.

Proposition 14. *Let R be a relation on a set X. For any two elements $x, y \in X$, it holds that*

(i) *If $x \approx_R y$, then it holds that*

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \begin{cases} [x]_{\approx_R} & \text{, if } x = y\,, \\ \{x, y\} & \text{, if } x \lhd_R y \vee x \rhd_R y\,, \\ \{x, y\} \cup \{z \in X \mid z \circ_R x \wedge z \circ_R y\} & \text{, if } x \circ_R y\,, \\ \{x, y\} \cup \{z \in X \mid z \diamond_R x \wedge z \diamond_R y\} & \text{, if } x \diamond_R y\,. \end{cases}$$

(ii) *If $x \not\approx_R y$ and $[x]_{\approx_R} \cap [y]_{\approx_R} \neq \emptyset$, then it holds that*

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{z\}\,,$$

where $z \in X$ is the unique element such that $x \rhd_R z$ and $z \rhd_R y$ or that $y \rhd_R z$ and $z \rhd_R x$.

Proof.

(i) Let $x, y \in X$ be such that $x \approx_R y$.
  (a) If $x = y$, then it trivially holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = [x]_{\approx_R} = [y]_{\approx_R}\,.$$

  (b) If $x \lhd_R y$, then we will prove that there does not exist any $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$ such that $z \in X \backslash \{x, y\}$. Suppose that such z exists. It then follows from Theorem 1 that $z \lhd_R x$ and $y \lhd_R z$, a contradiction (Corollary 9). Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{x, y\}\,.$$

  The proof is analogous for $x \rhd_R y$.

  (c) If $x \circ_R y$, then, for any $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$ such that $z \in X \backslash \{x, y\}$, it follows from Theorem 1 that $z \circ_R x$ and $z \circ_R y$. Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{x, y\} \cup \{z \in X \mid z \circ_R x \wedge z \circ_R y\}\,.$$

  (d) If $x \diamond_R y$, then, for any $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$ such that $z \in X \backslash \{x, y\}$, it follows from Theorem 1 that $x \diamond_R z$ and $y \diamond_R z$. Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{x, y\} \cup \{z \in X \mid z \diamond_R x \wedge z \diamond_R y\}\,.$$

(ii) Let $x, y \in X$ be such that $x \not\approx_R y$ and $[x]_{\approx_R} \cap [y]_{\approx_R} \neq \emptyset$ and let $z \in [x]_{\approx_R} \cap [y]_{\approx_R}$. It follows from Proposition 13 that z is the unique element such that $x \approx_R z$, $y \approx_R z$ and that $x \not\approx_R y$. Hence, it holds that

$$[x]_{\approx_R} \cap [y]_{\approx_R} = \{z\}\,.$$

□

Example 8. Let R be the relation defined in Example 1. It holds that $[a]_{\approx_R} = \{a, d\}, [b]_{\approx_R} = \{b, c\}, [c]_{\approx_R} = \{b, c\}, [d]_{\approx_R} = \{a, d\}, [e]_{\approx_R} = \{e, f\}$,
$[f]_{\approx_R} = \{e, f\}$. Therefore, it holds that:

$$[a]_{\approx_R} \cap [d]_{\approx_R} = \{a, d\}\,,$$
$$[b]_{\approx_R} \cap [c]_{\approx_R} = \{b, c\}\,,$$
$$[e]_{\approx_R} \cap [f]_{\approx_R} = \{e, f\}\,,$$
$$[e]_{\approx_R} \cap [d]_{\approx_R} = \emptyset\,.$$

Example 9. Let R be the relation defined in Example 2. For any $n \in \mathbb{N}^*$ with $n \neq 1$, it holds that $[n]_{\approx_<} = \{n-1, n, n+1\}$ ($[1]_{\approx_<} = \{1, 2\}$). As, for any $n_1, n_2 \in \mathbb{N}^*$, the fact that $[n_1]_{\approx_<} \cap [n_2]_{\approx_<} \neq \emptyset$ and that $n_1 \not\approx_R n_2$ implies that $n_1 = n_2 + 2$ or that $n_1 = n_2 - 2$, it follows that:

$$[n_1]_{\approx_<} \cap [n_2]_{\approx_<} = \begin{cases} n_2 - 1 & \text{, if } n_2 > n_1\,, \\ n_2 + 1 & \text{, if } n_1 > n_2\,. \end{cases}$$

5. The clone relation and the different types of disjoint union

In this section, we characterize the clone relation of the three different types of union of two relations defined on disjoint sets.

For a relation R_P defined on a set P, we write $\mathbb{P} = (P, R_P)$ and we call $\mathbb{P}$ an equipped set.

Definition 4. An equipped set $\mathbb{P} = (P, R_P)$ is called a reduction of another equipped set $\mathbb{Q} = (Q, R_Q)$ if the following two statements hold:

(i) $P \subseteq Q$.
(ii) For any $x, y \in P$, it holds that xR_Py if and only if xR_Qy.

If an equipped set is a reduction of another equipped set, then the clone relation of the second one is included in that of the first, as can be seen in the following proposition.

Proposition 15. *Let $\mathbb{P} = (P, R_P)$ be a reduction of $\mathbb{Q} = (Q, R_Q)$. For any $x, y \in P$, it holds that $x \approx_{R_Q} y$ implies that $x \approx_{R_P} y$.*

Proof. Let $x, y \in P$ be such that $x \approx_{R_Q} y$. It holds that $(zR_Qx \Leftrightarrow zR_Qy)$ and $(xR_Qz \Leftrightarrow yR_Qz)$, for any $z \in Q\backslash\{x, y\}$. Since $\mathbb{P} = (P, R_P)$ is a reduction of $\mathbb{Q} = (Q, R_Q)$, it follows that, for any $z \in P\backslash\{x, y\}$, it holds that $(zR_Px \Leftrightarrow zR_Py)$ and $(xR_Pz \Leftrightarrow yR_Pz)$. Hence, it holds that $x \approx_P y$. □

Remark 6. Note that, throughout this section, $\approx_{R_P}$ should be understood as the clone relation of R_P in P and not in $P \cup Q$. The same applies to $\approx_{R_Q}$.

Note that the converse of the statement in Proposition 15 does not hold, as can be seen in Example 10.

Example 10. Let us consider the sets $P = \mathbb{N}$ and $Q = \mathbb{R}$ equipped with the usual strict order relation $<$. It obviously holds that $\mathbb{P} = (\mathbb{N}, <_{\mathbb{N}})$ is a reduction of $\mathbb{Q} = (\mathbb{R}, <_{\mathbb{R}})$. However, it holds that $1 \approx_{<_{\mathbb{N}}} 2$, while $1 \not\approx_{<_{\mathbb{R}}} 2$. Hence, if $x \approx_P y$ for some $x, y \in P$, then it does not necessarily hold that $x \approx_Q y$.

For any two equipped sets $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$, we say that $\mathbb{P}$ and $\mathbb{Q}$ are disjoint if P and Q are disjoint. The union of two disjoint equipped sets is called a disjoint union. There are three different types of disjoint union: the nondirectional disjoint union, the unidirectional disjoint union and the bidirectional disjoint union.

The most common disjoint union is the nondirectional disjoint union, where the relations between elements in the same original set are kept and elements in different original sets are considered incomparable.

Definition 5. Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The nondirectional disjoint union $\mathbb{P} \cup \mathbb{Q}$ of $\mathbb{P}$ and $\mathbb{Q}$ is the equipped set $\mathbb{P} \cup \mathbb{Q} = (P \cup Q, R_P \cup R_Q)$.

The unidirectional disjoint union[4] is the disjoint union where the relations between elements in the same original set are kept and, for any element x in the first equipped set and any element y in the second equipped set, it holds that x is related with y but y is not related with x.

Definition 6. Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The unidirectional disjoint union of $\mathbb{P}$ and $\mathbb{Q}$ is the equipped set $\mathbb{P} \overrightarrow{\cup} \mathbb{Q} = (P \cup Q, R_P \overrightarrow{\cup} R_Q)$, where

$$R_P \overrightarrow{\cup} R_Q = R_P \cup R_Q \cup (P \times Q)\,.$$

[4] Note that, if $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ are two disjoint posets, then the unidirectional disjoint union of $\mathbb{P}$ and $\mathbb{Q}$ is known as the linear sum $\mathbb{P} \oplus \mathbb{Q}$ (see [3]).

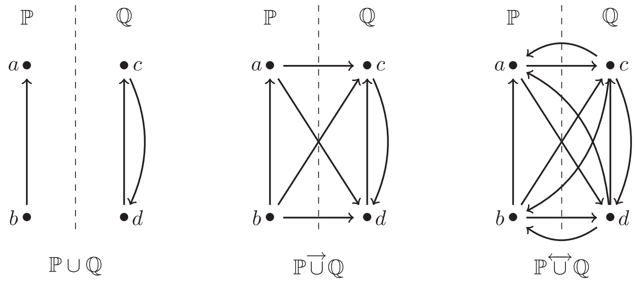

Fig. 6. Graphs of the three different types of disjoint union of two disjoint equipped sets.

The bidirectional disjoint union is the disjoint union where the relations between elements in the same original set are kept and, for any element x in the first equipped set and any element y in the second equipped set, it holds that x is related with y and y is related with x.

Definition 7. Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The bidirectional disjoint union of $\mathbb{P}$ and $\mathbb{Q}$ is the equipped set $\mathbb{P} \overleftrightarrow{\cup} \mathbb{Q} = (P \cup Q, R_P \overleftrightarrow{\cup} R_Q)$, where

$$R_P \overleftrightarrow{\cup} R_Q = R_P \cup R_Q \cup (P \times Q) \cup (Q \times P)\,.$$

Remark 7. Both the nondirectional disjoint union and the bidirectional disjoint union are commutative, while the unidirectional disjoint union is not.

The three types of disjoint union are illustrated in the following example.

Example 11. Let $P = \{a, b\}$, $Q = \{c, d\}$, $R_P = \{(b, a)\}$ and $R_Q = \{(c, d), (d, c)\}$. The graphs of the three different types of disjoint union are shown in Fig. 6 .

Now we characterize the clone relation of the three different types of disjoint union of two disjoint equipped sets. First, the clone relation of the nondirectional disjoint union is characterized.

Proposition 16. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The clone relation $\approx_R$ of the nondirectional disjoint union $R = R_P \cup R_Q$ is given by*

$$\approx_R = \approx_{R_P} \cup \approx_{R_Q} \cup (P_\| \times Q_\|) \cup (Q_\| \times P_\|)\,,$$

where $P_\| = \{x \in P \mid (\forall y \in P \setminus \{x\})(x \|_{R_P} y)\}$ and $Q_\| = \{x \in Q \mid (\forall y \in Q \setminus \{x\})(x \|_{R_Q} y)\}$.

Proof.

(i) First, we prove that $\approx_{R_P} \cup \approx_{R_Q} \cup (P_\| \times Q_\|) \cup (Q_\| \times P_\|) \subseteq \approx_R$.

(a) Let $x, y \in P$ be such that $x \approx_{R_P} y$. By definition of the nondirectional disjoint union, it follows that, for any $z_Q \in Q$, $(z_Q R^c x \wedge z_Q R^c y)$ and $(x R^c z_Q \wedge y R^c z_Q)$. Therefore, it holds that $(z_Q R x \Leftrightarrow z_Q R y)$ and $(x R z_Q \Leftrightarrow y R z_Q)$. Since P and Q are disjoint sets and $x \approx_{R_P} y$, it follows that $x, y \notin Q$ and, for any $z_P \in P \setminus \{x, y\}$, $(z_P R_P x \Leftrightarrow z_P R_P y)$ and $(x R_P z_P \Leftrightarrow y R_P z_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \cup \mathbb{Q}$, it follows that, for any $z \in (P \cup Q) \setminus \{x, y\}$, $(zRx \Leftrightarrow zRy)$ and $(xRz \Leftrightarrow yRz)$. Hence, it holds that $x \approx_R y$, and, thus, $\approx_{R_P} \subseteq \approx_R$. In an analogous way, we can prove that $\approx_{R_Q} \subseteq \approx_R$.

(b) Let $x \in P$ and $y \in Q$ be such that $(x, y) \in (P_\| \times Q_\|)$. On the one hand, by definition of $P_\|$ and $Q_\|$, it holds that for any $z \in (P \cup Q) \setminus \{x, y\}$, $z \|_{R_P} x$ and $z \|_{R_Q} y$. On the other hand, by definition of the nondirectional disjoint union, it holds that $z \|_{R_P} y$ and $z \|_{R_Q} x$. Therefore, it follows that $z \|_R x$ and $z \|_R y$, for any $z \in (P \cup Q) \setminus \{x, y\}$. This implies that $x \approx_R y$. Hence, it holds that $(P_\| \times Q_\|) \subseteq \approx_R$. In an analogous way, we can prove that $(Q_\| \times P_\|) \subseteq \approx_R$.

(ii) Second, we prove that $\approx_R \subseteq \approx_{R_P} \cup \approx_{R_Q} \cup (P_\| \times Q_\|) \cup (Q_\| \times P_\|)$.
Let $x, y \in P \cup Q$ be such that $x \approx_R y$. There are four cases to consider: ($x \in P$ and $y \in P$) or ($x \in Q$ and $y \in Q$) or ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).

(a) If $x, y \in P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \cup \mathbb{Q}$ and $x \approx_R y$, it follows from Proposition 15 that $x \approx_{R_P} y$.

(b) If $x, y \in Q$, then, again from Proposition 15, it follows that $x \approx_{R_Q} y$.

(c) If $x \in P$ and $y \in Q$, then one of the following cases holds: ($x \in P \setminus P_\|$ and $y \in Q$) or ($x \in P$ and $y \in Q \setminus Q_\|$) or ($x \in P_\|$ and $y \in Q_\|$). We will show that the two first cases lead to a contradiction.

(α) Suppose that $x \in P \setminus P_\|$ and $y \in Q$, then there exists $z \in P \setminus \{x\}$ such that $z R_P x$ or $x R_P z$. This implies that zRx or xRz. Since $y \in Q$, it follows that $z \|_R y$. From (zRx or xRz) and $z \|_R y$, it follows that $x \not\approx_R y$, a contradiction.

(β) Suppose that $x \in P$ and $y \in Q \setminus Q_\|$, then as in (α), it follows that $x \not\approx_R y$, a contradiction.

(d) If $x \in Q$ and $y \in P$, it follows analogously to (c) that $(x, y) \in Q_\| \times P_\|$. □

Corollary 11. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The partition $(\lhd_R, \circ_R, \diamond_R)$ of the clone relation $\approx_R$ of the nondirectional disjoint union $R = R_P \cup R_Q$ is given by:*

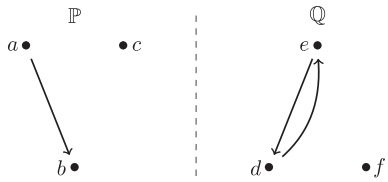

Fig. 7. Graphs of the relations R_P and R_Q.

(i) $\triangleleft_R = \triangleleft_{R_P} \cup \triangleleft_{R_Q}$;
(ii) $\circ_R = \circ_{R_P} \cup \circ_{R_Q}$;
(iii) $\diamond_R = \diamond_{R_P} \cup \diamond_{R_Q} \cup (P_\| \times Q_\|) \cup (Q_\| \times P_\|)$.

These results are illustrated in the following example.

Example 12. Let $P = \{a, b, c\}$, $Q = \{d, e, f\}$, $R_P = \{(a, b)\}$ and $R_Q = \{(e, d), (d, e)\}$. The graphs of the relations R_P and R_Q are shown in Fig. 7.

The matrix representations of $\approx_{R_P}$ and $\approx_{R_Q}$ are given by:

$$\approx_{R_P}= \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad \approx_{R_Q}= \begin{array}{c} \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

In addition, the matrix representation of the clone relation $\approx_{R_P \cup R_Q}$ of the nondirectional disjoint union $R_P \cup R_Q$ is given by:

$$\approx_{R_P \cup R_Q}= \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Note that $P_\| = \{c\}$ and $Q_\| = \{f\}$ and, therefore, it holds that $(P_\| \times Q_\|) \cup (Q_\| \times P_\|) = \{(c, f), (f, c)\}$.

Next, the clone relation of the disjoint unidirectional union is characterized.

Proposition 17. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The clone relation $\approx_R$ of the unidirectional disjoint union $R = R_P \overrightarrow{\cup} R_Q$ is given by*

$$\approx_R = \approx_{R_P} \cup \approx_{R_Q} \cup (P_\rightarrow \times Q_\leftarrow) \cup (Q_\leftarrow \times P_\rightarrow),$$

where $P_\rightarrow = \{x \in P \mid (\forall z_P \in P \setminus \{x\})(z_P R_P x \wedge x R_P^c z_P)\}$ and $Q_\leftarrow = \{y \in Q \mid (\forall z_Q \in Q \setminus \{y\})(y R_Q z_Q \wedge z_Q R_Q^c y)\}$[5].

Proof.

(i) First, we prove that $\approx_{R_P} \cup \approx_{R_Q} \cup (P_\rightarrow \times Q_\leftarrow) \cup (Q_\leftarrow \times P_\rightarrow) \subseteq \approx_R$.

(a) Let $x, y \in P$ be such that $x \approx_{R_P} y$. By definition of the unidirectional disjoint union, it follows that, for any $z_Q \in Q$, $(z_Q R^c x \wedge z_Q R^c y)$ and $(x R z_Q \wedge y R z_Q)$. Therefore, it holds that $(z_Q R x \Leftrightarrow z_Q R y)$ and $(x R z_Q \Leftrightarrow y R z_Q)$. Since P and Q are disjoint sets and $x \approx_{R_P} y$, it follows that $x, y \notin Q$ and, for any $z_P \in P \setminus \{x, y\}$, $(z_P R_P x \Leftrightarrow z_P R_P y)$ and $(x R_P z_P \Leftrightarrow y R_P z_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \overrightarrow{\cup} \mathbb{Q}$, it follows that, for any $z \in (P \cup Q) \setminus \{x, y\}$, $(zRx \Leftrightarrow zRy)$ and $(xRz \Leftrightarrow yRz)$. Hence, it holds that $x \approx_R y$, and, thus, $\approx_{R_P} \subseteq \approx_R$. In an analogous way, we can prove that $\approx_{R_Q} \subseteq \approx_R$.

(b) Let $x \in P$ and $y \in Q$ be such that $(x, y) \in P_\rightarrow \times Q_\leftarrow$. Let $z \in (P \cup Q) \setminus \{x, y\}$.

(α) If zRx, then, by definition of unidirectional disjoint union, it must hold that $z \in P$. It follows that $(z, y) \in P \times Q$ and, therefore, zRy.

(β) If zRy, then, since $y \in Q_\leftarrow$, it must hold that $z \in P$. Since $x \in P_\rightarrow$, it follows that $z R_P x$, and, therefore, zRx.

(γ) If xRz, then, since $x \in P_\rightarrow$, it must hold that $z \in Q$. Since $y \in Q_\leftarrow$, it follows that $y R_Q z$, and, therefore, yRz.

(δ) If yRz, then, by definition of unidirectional disjoint union, it must hold that $z \in Q$. It follows that $(x, z) \in P \times Q$ and, therefore, xRz.

Hence, it holds that $x \approx_R y$, and, thus, $P_\rightarrow \times Q_\leftarrow \subseteq \approx_R$. In an analogous way, we can prove that $Q_\leftarrow \times P_\rightarrow \subseteq \approx_R$.

[5] Note that both $P_\rightarrow$ and $Q_\leftarrow$ are either the empty set or a singleton.

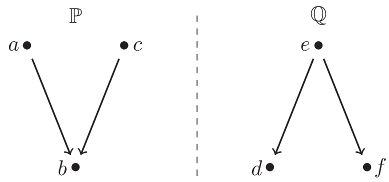

Fig. 8. Graphs of the relations R_P and R_Q.

(ii) Second, we prove that $\approx_R \subseteq \approx_{R_P} \cup \approx_{R_Q} \cup (P_\to \times Q_\leftarrow) \cup (Q_\leftarrow \times P_\to)$.
Let $x, y \in P \cup Q$ be such that $x \approx_R y$. There are four cases to consider: ($x \in P$ and $y \in P$), ($x \in Q$ and $y \in Q$), ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).
   (a) If $x, y \in P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \overrightarrow{\cup} \mathbb{Q}$ and $x \approx_R y$, it follows from Proposition 15 that $x \approx_{R_P} y$.
   (b) If $x, y \in Q$, then, again from Proposition 15, it follows that $x \approx_{R_Q} y$.
   (c) If $x \in P$ and $y \in Q$, then, on the one hand, for any $z_Q \in Q$, it follows that xRz_Q and z_QR^cx. Since $x \approx_R y$, it holds that yRz_Q and z_QR^cy, for any $z_Q \in Q \backslash \{y\}$. Hence, it holds that $y \in Q_\leftarrow$. On the other hand, for any $z_P \in P$, it holds that z_PRy and yR^cz_P. Since $x \approx_R y$, it follows that z_PRx and xR^cz_P, for any $z_P \in P \backslash \{x\}$. Hence, it holds that $x \in P_\to$. Thus, it holds that $(x, y) \in P_\to \times Q_\leftarrow$.
   (d) If $x \in Q$ and $y \in P$, it follows analogously to (c) that $(x, y) \in Q_\leftarrow \times P_\to$. □

Corollary 12. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The partition $(\triangleleft_R, \circ_R, \diamond_R)$ of the clone relation $\approx_R$ of the unidirectional disjoint union $R = R_P \overrightarrow{\cup} R_Q$ is given by:*

(i) $\triangleleft_R = \triangleleft_{R_P} \cup \triangleleft_{R_Q} \cup (P_\to \times Q_\leftarrow)$.
(ii) $\circ_R = \circ_{R_P} \cup \circ_{R_Q}$.
(iii) $\diamond_R = \diamond_{R_P} \cup \diamond_{R_Q}$.

These results are illustrated in the following example.

Example 13. Let $P = \{a, b, c\}$, $Q = \{d, e, f\}$, $R_P = \{(a, b), (c, b)\}$ and $R_Q = \{(e, d), (e, f)\}$. The graphs of the relations R_P and R_Q are shown in Fig. 8.

The matrix representations of $\approx_{R_P}$ and $\approx_{R_Q}$ are given by:

$$\approx_{R_P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \!\! \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \qquad \approx_{R_Q} = \begin{array}{c} \\ d \\ e \\ f \end{array} \!\! \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}.$$

In addition, the matrix representation of the clone relation $\approx_{R_P \overrightarrow{\cup} R_Q}$ of the unidirectional disjoint union $R_P \overrightarrow{\cup} R_Q$ is given by:

$$\approx_{R_P \overrightarrow{\cup} R_Q} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \!\! \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Note that $P_\to = \{b\}$ and $Q_\leftarrow = \{e\}$ and, therefore, it holds that $(P_\to \times Q_\leftarrow) \cup (Q_\leftarrow \times P_\to) = \{(b, e), (e, b)\}$.

As the unidirectional disjoint union is not commutative, we also analyse the unidirectional disjoint union $R_Q \overrightarrow{\cup} R_P$. The matrix representation of the clone relation $\approx_{R_Q \overrightarrow{\cup} R_P}$ of the unidirectional disjoint union $R_Q \overrightarrow{\cup} R_P$ is given by:

$$\approx_{R_Q \overrightarrow{\cup} R_P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \!\! \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

One may note that $\approx_{R_P \overrightarrow{\cup} R_Q}$ and $\approx_{R_Q \overrightarrow{\cup} R_P}$ do not coincide. For instance, it holds that $b \approx_{R_P \overrightarrow{\cup} R_Q} e$ but $b \not\approx_{R_Q \overrightarrow{\cup} R_P} e$.

We finish this subsection by characterizing the clone relation of the bidirectional disjoint union.

Proposition 18. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The clone relation $\approx_R$ of the bidirectional disjoint union $R = R_P \overset{\leftrightarrow}{\cup} R_Q$ is given by*

$$\approx_R = \approx_{R_P} \cup \approx_{R_Q} \cup (P_\leftrightarrow \times Q_\leftrightarrow) \cup (Q_\leftrightarrow \times P_\leftrightarrow)\,,$$

where $P_\leftrightarrow = \{x \in P \mid (\forall z_P \in P \setminus \{x\})(xR_Pz_P \wedge z_PR_Px)\}$ and $Q_\leftrightarrow = \{y \in Q \mid (\forall z_Q \in Q \setminus \{y\})(yR_Qz_Q \wedge z_QR_Qy)\}$.

Proof.

(i) First, we prove that $\approx_{R_P} \cup \approx_{R_Q} \cup (P_\leftrightarrow \times Q_\leftrightarrow) \cup (Q_\leftrightarrow \times P_\leftrightarrow) \subseteq \approx_R$.
  (a) Let $x, y \in P$ be such that $x \approx_{R_P} y$. By definition of the bidirectional disjoint union, it follows that, for any $z_Q \in Q$, $(z_QRx \wedge z_QRy)$ and $(xRz_Q \wedge yRz_Q)$. Therefore, it holds that $(z_QRx \Leftrightarrow z_QRy)$ and $(xRz_Q \Leftrightarrow yRz_Q)$. Since P and Q are disjoint sets and $x \approx_{R_P} y$, it follows that $x, y \notin Q$ and, for any $z_P \in P\backslash\{x, y\}$, $(z_PR_Px \Leftrightarrow z_PR_Py)$ and $(xR_Pz_P \Leftrightarrow yR_Pz_P)$. As $\mathbb{P}$ is a reduction of $\mathbb{P} \overset{\leftrightarrow}{\cup} \mathbb{Q}$, it follows that, for any $z \in (P \cup Q)\backslash\{x, y\}$, $(zRx \Leftrightarrow zRy)$ and $(xRz \Leftrightarrow yRz)$. Hence, it holds that $x \approx_R y$, and, thus, $\approx_{R_P} \subseteq \approx_R$. In an analogous way, we prove that $\approx_{R_Q} \subseteq \approx_R$.
  (b) Let $x \in P$ and $y \in Q$ be such that $(x, y) \in P_\leftrightarrow \times Q_\leftrightarrow$. Let $z \in (P \cup Q)\backslash\{x, y\}$.
    (α) If zRx, then we distinguish two cases: $z \in Q$ or $z \in P$. If $z \in Q$, then, by definition of $Q_\leftrightarrow$, it follows that zR_Qy. Hence, it holds that zRy. If $z \in P$, then it holds that $(z, y) \in P \times Q$ and, therefore, zRy.
    (β) If zRy, then we distinguish two cases: $z \in Q$ or $z \in P$. If $z \in Q$, then it holds that $(z, x) \in Q \times P$ and, hence, zRx. If $z \in P$, then by definition of $P_\leftrightarrow$, it follows that zR_Px. Hence, it holds that zRx.
    (γ) If xRz, then we prove in an analogous way to (α) that yRz.
    (δ) If yRz, then we prove in an analogous way to (β) that xRz.
  Hence, it holds that $x \approx_R y$, and, thus, $P_\leftrightarrow \times Q_\leftrightarrow \subseteq \approx_R$. In an analogous way, we can prove that $Q_\leftrightarrow \times P_\leftrightarrow \subseteq \approx_R$.

(ii) Second, we prove that $\approx_R \subseteq \approx_{R_P} \cup \approx_{R_Q} \cup (P_\leftrightarrow \times Q_\leftrightarrow) \cup (Q_\leftrightarrow \times P_\leftrightarrow)$.
  Let $x, y \in P \cup Q$ be such that $x \approx_R y$. There are four cases to consider: ($x \in P$ and $y \in P$) or ($x \in Q$ and $y \in Q$) or ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).
  (a) If $x, y \in P$, then, since $\mathbb{P}$ is a reduction of $\mathbb{P} \overset{\leftrightarrow}{\cup} \mathbb{Q}$ and $x \approx_R y$, it follows from Proposition 15 that $x \approx_{R_P} y$.
  (b) If $x, y \in Q$, then, again from Proposition 15, it follows that $x \approx_{R_Q} y$.
  (c) If $x \in P$ and $y \in Q$, then, on the one hand, since $x \in P$, it follows, by definition of bidirectional disjoint union, that xRz_Q and z_QRx, for any $z_Q \in Q$. Since $x \approx_R y$, it follows that yRz_Q and z_QRy, for any $z_Q \in Q\backslash\{y\}$. Hence, it holds that $y \in Q_\leftrightarrow$. On the other hand, since $y \in Q$, it follows that yRz_P and z_PRy, for any $z_P \in P$. Since $x \approx_R y$, it follows that xRz_P and z_PRx, for any $z_P \in P\backslash\{x\}$. Hence, it holds that $x \in P_\leftrightarrow$. Thus, it holds that $(x, y) \in P_\leftrightarrow \times Q_\leftrightarrow$.
  (d) If $x \in Q$ and $y \in P$, it follows analogously to (c) that $(x, y) \in Q_\leftrightarrow \times P_\leftrightarrow$. □

Corollary 13. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The partition $(\lhd_R, \circ_R, \diamond_R)$ of the clone relation $\approx_R$ of the bidirectional disjoint union $R = R_P \overset{\leftrightarrow}{\cup} R_Q$ is given by:*

(i) $\lhd_R = \lhd_{R_P} \cup \lhd_{R_Q}$;
(ii) $\circ_R = \circ_{R_P} \cup \circ_{R_Q} \cup (P_\leftrightarrow \times Q_\leftrightarrow) \cup (Q_\leftrightarrow \times P_\leftrightarrow)$;
(iii) $\diamond_R = \diamond_{R_P} \cup \diamond_{R_Q}$.

These results are illustrated in the following example.

Example 14. Let $P = \{a, b, c\}$, $Q = \{d, e, f\}$, $R_P = \{(a, c), (c, a), (c, b), (b, c)\}$ and $R_Q = \{(d, e), (e, d), (d, f), (f, d), (e, f), (f, e)\}$. The graphs of the relations R_P and R_Q are shown in Fig. 9.

The matrix representations of $\approx_{R_P}$ and $\approx_{R_Q}$ are given by:

$$\approx_{R_P} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad \approx_{R_Q} = \begin{array}{c c} & \begin{array}{ccc} d & e & f \end{array} \\ \begin{array}{c} d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}.$$

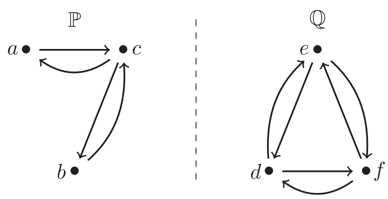

Fig. 9. Graphs of the relations R_P and R_Q.

In addition, the matrix representation of the clone relation $\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q}$ of the bidirectional disjoint union $R_P \overset{\leftrightarrow}{\cup} R_Q$ is given by:

$$\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Note that $P_{\leftrightarrow} = \{c\}$ and $Q_{\leftrightarrow} = \{d, e, f\}$ and, therefore, it holds that $(P_{\leftrightarrow} \times Q_{\leftrightarrow}) \cup (Q_{\leftrightarrow} \times P_{\leftrightarrow}) = \{(c, d), (c, e), (c, f), (d, c), (e, c), (f, c)\}$.

We conclude this section by discussing when the clone relation of the different types of disjoint union of R_P and R_Q coincide with the union of the clone relations of R_P and R_Q.

Theorem 2. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The following statements hold:*

(i) $\approx_{R_P \cup R_Q} = \approx_R \cup \approx_Q$ *if and only if* $(\forall x \in P)(\exists y \in P\setminus\{x\})(xR_Py \vee yR_Px) \vee (\forall x \in Q)(\exists y \in Q\setminus\{x\})(xR_Qy \vee yR_Qx)$.

(ii) $\approx_{R_P \overset{\rightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q$ *if and only if* $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_Py \vee yR_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Qx)$.

(iii) $\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q$ *if and only if* $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_P^c y \vee yR_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Q^c x)$.

Proof.

(i) Note that, due to Proposition 16, $(\approx_{R_P \cup R_Q} = \approx_R \cup \approx_Q)$ is equivalent to $(P_{\|} = \emptyset) \vee (Q_{\|} = \emptyset)$. Furthermore, it trivially follows that, by definition of $P_{\|}$ and $Q_{\|}$, $(P_{\|} = \emptyset) \vee (Q_{\|} = \emptyset)$ is equivalent to $(\forall x \in P)(\exists y \in P\setminus\{x\})(xR_Py \vee yR_Px) \vee (\forall x \in Q)(\exists y \in Q\setminus\{x\})(xR_Qy \vee yR_Qx)$.

(ii) Note that, due to Proposition 17, $(\approx_{R_P \overset{\rightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q)$ is equivalent to $(P_{\rightarrow} = \emptyset) \vee (Q_{\leftarrow} = \emptyset)$. Furthermore, it trivially follows that, by definition of $P_{\rightarrow}$ and $Q_{\leftarrow}$, $(P_{\rightarrow} = \emptyset) \vee (Q_{\leftarrow} = \emptyset)$ is equivalent to $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_Py \vee yR_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Qx)$.

(iii) Note that, due to Proposition 18, $(\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q)$ is equivalent to $(P_{\leftrightarrow} = \emptyset) \vee (Q_{\leftrightarrow} = \emptyset)$. Furthermore, it trivially follows that, by definition of $P_{\leftrightarrow}$ and $Q_{\leftrightarrow}$, $(P_{\leftrightarrow} = \emptyset) \vee (Q_{\leftrightarrow} = \emptyset)$ is equivalent to $(\forall x \in P)(\exists y \in P \setminus \{x\})(xR_P^c y \vee yR_P^c x) \vee (\forall x \in Q)(\exists y \in Q \setminus \{x\})(xR_Q^c y \vee yR_Q^c x)$. □

Corollary 14. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two disjoint equipped sets. The following statements hold:*

(i) *If either R_P or R_Q is complete, then* $\approx_{R_P \cup R_Q} = \approx_R \cup \approx_Q$.

(ii) *If either R_P or R_Q is symmetric, then* $\approx_{R_P \overset{\rightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q$.

(iii) *If either R_P or R_Q is antisymmetric, then* $\approx_{R_P \overset{\leftrightarrow}{\cup} R_Q} = \approx_R \cup \approx_Q$.

6. Conclusions and future research lines

In this work, we have extended the notion of clone relation of a strict order relation to an arbitrary binary relation. Throughout this paper, the basic properties of this clone relation have been analysed. We have also proposed a partition of the clone relation in terms of three different types of pairs of clones. One type of pairs of clones leads to an antitransitive relation, while both the two other types of pairs of clones lead to a transitive relation. This partition of the clone relation has not only been an important tool in the proofs of this paper, but it also helps to gain a deeper understanding of the structure of the clone relation and it will be a key element in future work. Finally, the clone relation of the three different types of disjoint union has been characterized.

Future work is anticipated in multiple directions. First, we will exploit the properties of the clone relation provided in this paper for characterizing the fuzzy tolerance relations or fuzzy equivalence relations that a binary relation is compatible with. Second, we will extend the clone relation of a binary relation to fuzzy relations. In this context, connections with the field of fuzzy preference modelling, in particular the study of additive fuzzy preference structures [4,14], will be explored. Third, sets of clones, which are a generalization of pairs of clones in the sense of this paper, have already been analysed by Tideman in the field of social choice theory for the special case of total order relations. These sets of clones will be extended to an arbitrary binary relation in the near future.

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