Completeness and Compactness in Standard Single Valued Neutrosophic Metric Spaces

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Abstract

In a recent paper, we have introduced the notion of standard single valued neutrosophic metric space as a generalization of standard fuzzy metric spaces given by J.R. Kider and Z.A. Hussain. In this paper, we continue our previous work by introducing the notions of complete standard single valued neutrosophic metric space and compact standard single valued neutrosophic metric space. Furthermore, we give a number of properties and characterizations of these notions and relationship between them.
Keywords: Single valued neutrosophic set, Metric space, Completeness, Compactness

1 Introduction

In 1995, Smarandache proposed the notion of neutrosophic set, which was published in 1998 [10] as a generalization of the notions of fuzzy set and intuitionistic fuzzy set. A neutrosophic set (NS) is a set where each element of the universe has a degree of truth (T), indeterminacy (I) and falsity (F) in the non standard unit interval. Further, Wang et al. [15] proposed the notion of single valued neutrosophic set (SVNS) as a subclass of (NS). Single valued neutrosophic sets have been useful in many real applications in several branches (see for e.g., [2,5,6,7,13] and [17]).
In the literature, there are several approaches to the notion of neutrosophic metric space. In [14], Taş et al. defined the neutrosophic valued metric spaces and neutrosophic valued g-metric spaces. In this regard, we find that other authors have adopted the same approach, such as Şahin et al. [8,9]. Later on, Kirişci and Şimşek [4] introduced neutrosophic metric space with neutrosophic numbers and they investigated some properties of neutrosophic metric space such as compactness and completeness. In the present study, we introduce the notion of standard single valued neutrosophic metric space and, from this notion, we introduce the notion of complete standard single valued neutrosophic metric space and compact standard single valued neutrosophic metric space. Furthermore, we give a number of properties and characterizations of these notions and relationship between them.
This paper is structured as follows. In Section 2, we recall basic concepts and properties of single valued neutrosophic sets. Moreover, we introduce the concept of standard single valued neutrosophic metric spaces and some related notions that will be needed throughout this paper. In Section 3, we introduce the notion of complete single valued neutrosophic metric space and we show its interesting properties. In Section 4, we introduce the notion of compact single valued neutrosophic metric space with interesting characterizations and properties and relationship between completeness and compactness. Finally, we present some conclusions and we discuss future research in Section 5.

2 Preliminaries

This section contains the basic definitions and properties of single valued neutrosophic sets and some related notions that will be needed throughout this paper.

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2.1 Single valued neutrosophic sets

The notion of fuzzy sets was first introduced by Zadeh [18].

Definition 2.1. [18] Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for any $x \in X$.

In 1983, Atanassov [1] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.2. [1] Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ which satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

In 1998, Smarandache [10] defined the concept of a neutrosophic set as a generalization of Atanassov's intuitionistic fuzzy set. Also, he introduced neutrosophic logic, neutrosophic set and its applications in [11,12]. In particular, Wang et al. [15] introduced the notion of a single valued neutrosophic set.

Definition 2.3. [11] Let X be a nonempty set. A neutrosophic set (NS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to]^-0, 1^+[$ and an indeterminacy function $\sigma_A : X \to]^-0, 1^+[$ and a non-membership function $\nu_A : X \to]^-0, 1^+[$ which satisfy the condition:

$$^-0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+, \text{ for any } x \in X.$$

Certainly, intuitionistic fuzzy sets are neutrosophic sets by setting $\sigma_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Next, we show the notion of single valued neutrosophic set as an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 2.4. [15] Let X be a nonempty set. A single valued neutrosophic set (SVNS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a truth-membership function $\mu_A : X \to [0, 1]$, an indeterminacy-membership function $\sigma_A : X \to [0, 1]$ and a falsity-membership function $\nu_A : X \to [0, 1]$.

The class of single valued neutrosophic sets on X is denoted by $SVN(X)$.

For any two SVNSs A and B on a set X, several operations are defined (see, e.g., [15,16]). Here we will present only those which are related to the present paper.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$,

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\sigma_A(x) = \sigma_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,

(v) $\overline{A} = \{\langle x, 1 - \nu_A(x), 1 - \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$.

2.2 Standard single valued neutrosophic metric spaces

In this Subsection, we extend the notion of standard fuzzy metric space introduced by J.R. Kider and Z.A. Hussain [3] to the setting of single valued neutrosophic sets. Also, we discuss the main properties related to this notion.

Definition 2.5. A quintuple $(X, M, *, \triangleleft, \diamond)$ is said to be a standard single valued neutrosophic metric space (SSVN-metric space, for short) if X is an arbitrary set, $*$, $\triangleleft$ are a continuous t-norms, $\diamond$ is a t-conorm and M is a continuous single valued neutrosophic set on X^2 satisfying the following conditions:

(i) $\mu_M(x, y) > 0$, $\sigma_M(x, y) > 0$ and $\nu_M(x, y) < 1$ for all $x, y \in X$;

(ii) $\mu_M(x, y) = 1$, $\sigma_M(x, y) = 1$ and $\nu_M(x, y) = 0$ if and only if $x = y$;

(iii) $\mu_M(x,y) = \mu_M(y,x),\ \sigma_M(x,y) = \sigma_M(y,x)$ and $\nu_M(x,y) = \nu_M(y,x)$ for all $x, y \in X$;

(iv) $\mu_M(x,z) \geq \mu_M(x,y) * \mu_M(y,z),\ \sigma_M(x,z) \geq \sigma_M(x,y) \triangleleft \sigma_M(y,z)$ and $\nu_M(x,z) \leq \nu_M(x,y) \diamond \nu_M(y,z)$.

The functions $\mu_M(x,y),\ \sigma_M(x,y)$ and $\nu_M(x,y)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between x and y, respectively.

Remark 2.6. If the set X given in the previous definition is a metric space with an ordinary distance d, then $(X, M, *, \triangleleft, \diamond)$ is called an SSVN-metric space induced by (X, d).

Example 2.7. Let (X, d) be an ordinary metric space. Define the t-norms $x * y = min\{x, y\},\ x \triangleleft y = min\{x, y\}$ and the t-conorm $x \diamond y = max\{x, y\}$, for all $x,\ y \in [0, 1]$. Define the single valued neutrosophic set M on X^2 as:
$\mu_M(x,y) = \frac{1}{1+d(x,y)},\ \sigma_M(x,y) = d(x,y),\ \nu_M(x,y) = \frac{d(x,y)}{1+d(x,y)}$.
Then, $(X, M, *, \triangleleft, \diamond)$ is an SSVN-metric space.

Definition 2.8. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. For $x \in X$ and $r \in]0, 1[$, the open ball $\mathcal{B}(x, r)$ with radius r and center x is defined by

$$\mathcal{B}(x, r) = \{y \in X \mid \mu_M(x,y) > 1 - r,\ \sigma_M(x,y) > 1 - r \text{ and } \nu_M(x,y) < r\}.$$

Definition 2.9. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space, a subset A of X is said to be an open set (OS, for short) if for any $x \in A$ there exists $r \in]0, 1[$ such that $\mathcal{B}(x, r) \subseteq A$. The complement of an open set is called a closed set (CS, for short) in X.

Definition 2.10. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. Then

(i) a sequence (x_n) in X is said to be convergent to a point $x \in X$ if for any $r \in]0, 1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(x_n, x) > 1 - r, \sigma_M(x_n, x) > 1 - r \text{ and } \nu_M(x_n, x) < r, \text{ for all } n \geq k\,.$$

(ii) a sequence (x_n) in X is said to be Cauchy sequence if for any $r \in]0, 1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(x_n, x_m) > 1 - r, \sigma_M(x_n, x_m) > 1 - r \text{ and } \nu_M(x_n, x_m) < r, \text{ for all } n, m \geq k\,.$$

Definition 2.11. Let $(X, M, *, \triangleleft, \diamond)$ ba an SSVN-metric space and let $A \subseteq X$ then the closure of A is denoted by $\overline{A}$ is defined by the set of all limit of sequences (x_n) in A.

Definition 2.12. Let $(X, M, *, \triangleleft, \diamond)$ ba an SSVN-metric space and let $A \subseteq X$. A is said to be dense in X if $\overline{A} = X$.

Remark 2.13. Let $r_1, r_2 \in [0, 1]$. If $r_1 > r_2$, then there exist $r_3, r_4 \in]0, 1[$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_2 \diamond r_4$. Moreover, for any $r_5 \in]0, 1[$, there exist $r_6, r_7 \in]0, 1[$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

3 Completeness in SSVN-metric spaces

In this section, we will study some interesting properties of completeness in single valued neutrosophic metric spaces. First, we introduce the notion of complete single valued neutrosophic metric space.

Definition 3.1. An SSVN-metric space in which every Cauchy sequence is convergent, is said to be complete.

Theorem 3.2. *If every Cauchy sequence in an SSVN-metric space $(X, M, *, \triangleleft, \diamond)$ has a convergent subsequences. Then $(X, M, *, \triangleleft, \diamond)$ is complete.*

Proof. Let (x_n) be a Cauchy sequence, and let (x_{i_n}) be a subsequence of (x_n) where x_{i_n} converges to x. Take $r \in]0, 1[$ such that

$$(1 - r) * (1 - r) > 1 - \alpha, (1 - r) \triangleleft (1 - r) > 1 - \alpha \text{ and } r \diamond r < \alpha, \text{ for all } \alpha \in]0, 1[\,.$$

Since x_{i_n} converges to x, there exists $i_p \in \mathbb{N}$ such that

$$\mu_M(x_{i_n}, x) > 1 - r,\ \sigma_M(x_{i_n}, x) > 1 - r \text{ and } \nu_M(x_{i_n}, x) < r, \text{ for all } i_n \geq i_p\,.$$

In fact that (x_n) is a Cauchy sequence, then there exists $k \in \mathbb{N}$ where $k \geq i_p$ such that

$$\mu_M(x_n, x_m) > 1 - r,\ \sigma_M(x_n, x_m) > 1 - r \text{ and } \nu_M(x_n, x_m) < r,\ \text{for all } n,\ m \geq k\,.$$

Therefore, if $n \geq i_p$, then

$$\mu_M(x_n, x) \geq \mu_M(x_n, x_m) * \mu_M(x_m, x) > (1 - r) * (1 - r) > 1 - \alpha,$$

$$\sigma_M(x_n, x) \geq \sigma_M(x_n, x_m) \triangleleft \sigma_M(x_m, x) > (1 - r) \triangleleft (1 - r) > 1 - \alpha$$

$$\text{and } \nu_M(x_n, x) \leq \nu_M(x_n, x_m) \diamond \nu_M(x_m, x) < r \diamond r < \alpha\,.$$

Thus, we have x_n converges to x, and hence $(X, M, *, \triangleleft, \diamond)$ is complete. □

Next, we discuss the relationship between dense subset and completeness in SSVN-metric space. First, we need to provide the following key result.

Lemma 3.3. *Let A be an SSVN-metric space $(X, M, *, \triangleleft, \diamond)$. If A is dense in X, then there exists $a \in A$ such that*

$$\mu_M(x, a) > 1 - r,\ \sigma_M(x, a) > 1 - r \text{ and } \nu_M(x, a) < r,\ \text{where } r \in]0, 1[\text{ and } x \in X\,.$$

Proof. Suppose that A is dense in X and let $x \in X$. Then, $x \in \overline{A}$, and hence there exists a sequence (a_n) in A such that a_n converges to x. Hence, for any $r \in]0, 1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(a_n, x) > 1 - r,\ \sigma_M(a_n, x) > 1 - r \text{ and } \nu_M(a_n, x) < r,\ \text{for all } n \geq k\,.$$

Now, if we take $a = a_k$, then

$$\mu_M(a, x) > 1 - r,\ \sigma_M(a, x) > 1 - r \text{ and } \nu_M(a, x) < r,\ \text{for all } k \geq N\,.$$

This is the desired result. □

Theorem 3.4. *Let A be a dense subset of an SSVN-metric space $(X, M, *, \triangleleft, \diamond)$. If every Cauchy sequence of points of A converges in X then $(X, M, *, \triangleleft, \diamond)$ is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. On the one hand, since A is dense, it follows from Lemma 3.3 that for every $x_n \in X$ there exists $a_n \in A$ such that

$$\mu_M(x_n, a_n) > 1 - s,\ \sigma_M(x_n, a_n) > 1 - s \text{ and } \nu_M(x_n, a_n) < s \text{ where } s \in]0, 1[\,.$$

On the other hand, from Remark 2.13 there exists $t = 1 - \varepsilon \in]0, 1[$, such that

$$(1 - s) * (1 - s) > t,\ (1 - s) \triangleleft (1 - s) > t \text{ and } s \diamond s < \varepsilon\,.$$

We next show that the sequence (a_n) is Cauchy.
Indeed, since (x_n) is Cauchy in X, it then follows that for any $r \in]0, 1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(x_n, x_m) > t,\ \sigma_M(x_n, x_m) > t \text{ and } \nu_M(x_n, x_m) < \varepsilon \text{ for all } n, m \geq k\,.$$

Therefore,

$$\mu_M(a_n, a_m) \geq \mu_M(a_n, x_n) * \mu_M(x_n, a_m) > (1 - s) * (1 - s) > t,$$

$$\sigma_M(a_n, a_m) \geq \sigma_M(a_n, x_n) \triangleleft \sigma_M(x_n, a_m) > (1 - s) \triangleleft (1 - s) > t$$

$$\text{and } \nu_M(a_n, a_m) \leq \nu_M(a_n, x_n) \diamond \nu_M(x_n, a_m) < s \diamond s < \varepsilon\,.$$

Then (a_n) is Cauchy sequence and since A is dense of $(X, M, *, \triangleleft, \diamond)$ this implies that (a_n) is converges to $x \in X$. On the other hand, $\mu_M(x_n, x) \geq \mu_M(x_n, a_n) * \mu_M(a_n, x) \geq (1 - s) * (1 - s) \geq 1 - \varepsilon$, $\sigma_M(x_n, x) \geq \sigma_M(x_n, a_n) \triangleleft \sigma_M(a_n, x) \geq (1 - s) \triangleleft (1 - s) \geq 1 - \varepsilon$ and $\nu_M(x_n, x) \leq \nu_M(x_n, a_n) \diamond \nu_M(a_n, x) \leq (s \diamond s) \leq \varepsilon$ Then (x_n) is converges to x. Hence, $(X, M, *, \triangleleft, \diamond)$ is complete. □

Now, we introduce the notion of continuous mapping and uniformly continuous mapping in SSVN-metric spaces.

Definition 3.5. Let $(X, M_X, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous at $a \in X$, if for every $r \in]0,1[$, there exists $\delta \in]0,1[$ such that

$$\mu_{M'}(f(x), f(a)) > 1 - r,\ \sigma_{M'}(f(x), f(a)) > 1 - r \text{ and } \nu_{M'}(f(x), f(a)) < r\,,$$

$$\text{whenever } \mu_M(x, a) > 1 - \delta,\ \sigma_M(x, a) > 1 - \delta \text{ and } \nu_M(x, a) < \delta\,.$$

Definition 3.6. Let $(X, M_X, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic uniformly continuous on X, if for every $r \in]0,1[$, there exists $\delta \in]0,1[$ such that

$$\mu_{M'}(f(x_1), f(x_2)) > 1 - r,\ \sigma_{M'}(f(x_1), f(x_2)) > 1 - r \text{ and } \nu_{M'}(f(x_1), f(x_2)) < r\,,$$

$$\text{whenever } \mu_M(x_1, x_2) > 1 - \delta,\ \sigma_M(x_1, x_2) > 1 - \delta \text{ and } \nu_M(x_1, x_2) < \delta\,.$$

Theorem 3.7. *Let $f : (X, M, *, \triangleleft, \diamond) \longrightarrow (Y, M', *, \triangleleft, \diamond)$ to be a one-to-one and uniformly continuous. If f^{-1} is a single valued neutrosophic continuous and Y is complete, then X is complete.*

Proof. Suppose (x_n) is a Cauchy sequence and let the sequence $y_n = f(x_n)$. We show that (y_n) is a Cauchy sequence. Since (x_n) is a Cauchy sequence, it follows that

$$\mu_M(x_1, x_2) > 1 - \delta,\ \sigma_M(x_1, x_2) > 1 - \delta \text{ and } \nu_M(x_1, x_2) < \delta\,,$$

for any $\delta \in]0,1[$. This implies that

$$\mu_{M'}(f(x_1), f(x_2)) > 1 - r,\ \sigma_{M'}(f(x_1), f(x_2)) > 1 - r \text{ and } \nu_{M'}(f(x_1), f(x_2)) < r\,,$$

for any $r \in]0,1[$ and, there exists $k \in \mathbb{N}$ such that $m, n > k$ imply that

$$\mu_M(x_n, x_m) > 1 - \delta,\ \sigma_M(x_n, x_m) > 1 - \delta \text{ and } \nu_M(x_n, x_m) < \delta\,.$$

It follows that for $m, n > k$

$$\mu_{M'}(y_n, y_m) > 1 - r,\ \sigma_{M'}(y_n, y_m) > 1 - r \text{ and } \nu_{M'}(y_n, y_m) < r\,.$$

Hence, (y_n) is Cauchy sequence which implies that there exists a subsequence (y_{n_k}) such that y_{n_k} converge to y, where $y \in Y$. Since f^{-1} is a single valued neutrosophic continuous mapping, it follows that $x_{n_k} = f^{-1}(y_{n_k})$ converges to $f^{-1}(y) = x$. According to Theorem 3.2, X is complete. □

4 Compactness in SSVN-metric spaces

In this section, we will study some interesting properties and characterizations of compactness in single valued neutrosophic metric spaces.

4.1 Definitions

In this subsection, we introduce the notion of SVN-bounded subset, totally bounded subset and compact single valued neutrosophic metric space.

Definition 4.1. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. A is said to be an SVN-bounded if there exists $r \in]0,1[$ such that $\mu_M(x, y) > 1 - r,\ \sigma_M(x, y) > 1 - r$ and $\nu_M(x, y) < r$, for all $x, y \in A$.

Definition 4.2. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$.
A collection $\mathcal{O}$ of open sets is called an open cover of A if, $A \subseteq \underset{U \in \mathcal{O}}{\cup} U$.

Definition 4.3. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. A is said to be a totally bounded if there exists $r \in]0,1[$ such that $\mu_M(x, y_i) > 1 - r,\ \sigma_M(x, y_i) > 1 - r$ and $\nu_M(x, y_i) < r$, for all $x \in X$ and $y_i \in A$ with $i = 1, ..., n$.

Definition 4.4. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space.
X is said to be compact if, $X = \overset{n}{\underset{i=1}{\cup}} U_i \mid U_i \subseteq \mathcal{O}$. In other words, if every open cover has a finite subcover.

4.2 Characterizations of compact SSVN-metric spaces

In this subsection, we provide interesting characterizations of compact SSVN-metric spaces.

Proposition 4.5. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. If A is a totally bounded, then A is an SVN-bounded.*

Proof. Assume that A is a totally bounded subset of X, and consider an open cover $\{\mathcal{B}(x,r), x \in A\}$ of A. Since A is a totally bounded, there exist $x_1, x_2, ..., x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} \mathcal{B}(x_i, r)$. Let $x, y \in A$, then $x \in \mathcal{B}(x_i, r)$ and $y \in \mathcal{B}(x_j, r)$, for some $1 \leq i, j \leq n$. Consequently,

$$\mu_M(x_i, x) > 1-r,\ \sigma_M(x_i, x) > 1-r \text{ and } \nu_M(x_i, x) < r$$

$$\text{and } \mu_M(x_j, y) > 1-r,\ \sigma_M(x_j, y) > 1-r \text{ and } \nu_M(x_j, y) < r.$$

Due to the symmetry of the functions $\mu_M,\ \sigma_M$ and ν_M (see (iii) of Definition 2.5), it holds that

$$\mu_M(x, x_i) > 1-r,\ \sigma_M(x, x_i) > 1-r \text{ and } \nu_M(x, x_i) < r$$

$$\text{and } \mu_M(y, x_j) > 1-r,\ \sigma_M(y, x_j) > 1-r \text{ and } \nu_M(y, x_j) < r.$$

Now, we put $r_0 = \min\{\mu_M(x, x_i); 1 \leq i, j \leq n\}$, $r_1 = \min\{\sigma_M(x, x_i); 1 \leq i, j \leq n\}$ and $r_2 = \max\{\nu_M(x, x_i); 1 \leq i, j \leq n\}$. Then there exists $s \in]0,1[$ such that $r_0 > 1-s > 1-r$, $r_1 > 1-s > 1-r$ and $r_2 < s < r$. Moreover, we obtain

$$\mu_M(x, y) \geq \mu_M(x, x_i) * \mu_M(x_i, x_j) * \mu_M(x_j, x) \geq (1-r) * (1-r) * r_0 > 1-s > 1-r,$$

$$\sigma_M(x, y) \geq \sigma_M(x, x_i) \triangleleft \sigma_M(x_i, x_j) \triangleleft \sigma_M(x_j, x) \geq (1-r) \triangleleft (1-r) \triangleleft r_1 > 1-s > 1-r$$

$$\text{and } \nu_M(x, y) \leq \nu_M(x, x_i) \diamond \nu_M(x_i, x_j) \diamond \nu_M(x_j, x) \leq r \diamond r \diamond r_2 < s < r.$$

Therefore,

$$\mu_M(x, y) > 1-r,\ \sigma_M(x, y) > 1-r \text{ and } \nu_M(x, y) < r \text{ for all } x, y \in A.$$

Hence, A is an SVN-bounded. □

Proposition 4.6. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. If X is compact, then X is totally bounded.*

Proof. It is clear that for any given $r \in]0,1[$, the collection $\mathcal{O}$ of all balls $\mathcal{B}(x,r)$ is an open cover of X, where $x \in X$. Let X be a compact SSVN-metric space. Since X is compact, it follows that $\mathcal{O}$ contains a finite subcover. Hence, for $r \in]0,1[$, there exists a finite number of open balls $\mathcal{B}(x_i, r)$ which represents an open cover of X, where $i = 1, 2, ..., n$. Now, if we consider $x \in \mathcal{B}(x_i, r)$ then

$$\mu_M(x, x_i) > 1-r,\ \sigma_M(x, x_i) > 1-r \text{ and } \nu_M(x, x_i) < r, \text{ for } i = 1, 2, ..., n.$$

Therefore, X is totally bounded. □

Combining Proposition 4.6 and Proposition 4.5 easily leads to the following result.

Corollary 4.7. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. If A is compact, then A is an SVN-bounded.*

Proposition 4.8. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. If X is compact, then X is complete.*

Proof. Suppose that X is compact and consider at the same time that X is not complete. Since X is not complete, there exists a Cauchy sequence (x_n) not having a limit in X. Now, we assume that $x \in X$. Since (x_n) does not converge to x, there exists $r_1 \in]0,1[$ such that

$$\mu_M(x_n, x) \leq 1-r_1,\ \sigma_M(x_n, x) \leq 1-r_1 \text{ and } \nu_M(x_n, x) \geq r_1 \text{ for all } x, y \in A \text{ for any } n \in \mathbb{N}.$$

In addition, as long as (x_n) is Cauchy, there exists an integer $k \in \mathbb{N}$ such that $n, m \geq k$. This implies that

$$\mu_M(x_n, x_m) > 1-r_2,\ \sigma_M(x_n, x_m) > 1-r_2 \text{ and } \nu_M(x_n, x_m) < r_2, \text{ where } r_2 \in]0,1[.$$

Next, we choose $m \geq k$ for which

$$\mu_M(x_m, x) > 1 - r_2,\ \sigma_M(x_m, x) > 1 - r_2 \text{ and } \nu_M(x_m, x) < r_2.$$

Then, the open ball $\mathcal{B}(x, r_2)$ contains x_n for only a finite number of values of $n \geq k$. Thus, it can be observed that $X = \overset{n}{\underset{x\in X}{\cup}} \mathcal{B}(x, r)$, which means that $\mathcal{O} = \{\mathcal{B}(x, r), x \in X\}$ is an open cover of X. The compactness of X implies that this open cover contains a finite subcover $\mathcal{O}_i = \{\mathcal{B}(x_i, r_i), x_i \in X, i = 1, 2, ..., n\}$.
In conclusion, as each ball contains (x_n) for only a finite number of values of $n \geq k$, the balls are in $\mathcal{O}_i$. In addition, X must contains (x_n) also for only a finite number of values of $n \geq k$ which means that a Cauchy sequence (x_n) have a limit in X. This, contradicts the hypothesis. Hence X is complete. □

Proposition 4.9. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. If X is totally bounded and complete, then X is compact.*

Proof. Suppose that X is totally bounded and complete and consider at the same time that X is not compact. On the one hand, since X is not compact, then there exists an open cover $U_i, i \in I$ of X which does not contain a finite subcover. On the other hand, since X is totally bounded, then it follows from Proposition 4.5 that $X \subseteq \mathcal{B}(x, r)$, for any $x \in X$ and $r \in]0, 1[$. While it is clear that $\mathcal{B}(x, r) \subseteq X$, this implies that $X = \mathcal{B}(x, r)$. Now, Setting $\alpha_n = \frac{r}{2^n}$. According to what we know that X being totally bounded can be covered by finite many balls of radius α_1, then from our hypothesis at least one of these balls, and so be it $\mathcal{B}(x_1, \alpha_1)$, cannot be covered by a finite number of sets U_i. As $\mathcal{B}(x_1, \alpha_1)$ is totally bounded, then we can find an $x_2 \in \mathcal{B}(x_1, \alpha_1)$ such that $\mathcal{B}(x_2, \alpha_2)$ cannot be covered by a finite number of sets U_i. Proceeding in this way, a sequence (x_n) can be defined with the property that for each n, $\mathcal{B}(x_n, \alpha_n)$ cannot be covered by a finite number of sets U_i and $x_{n+1} \in \mathcal{B}(x_n, \alpha_n)$.
Next, we show that the sequence (x_n) is convergent. The fact that $x_{n+1} \in \mathcal{B}(x_n, \alpha_n)$ implies that

$$\mu_M(x_n, x_{n+1}) > 1 - \alpha_n,\ \sigma_M(x_n, x_{n+1}) > 1 - \alpha_n \text{ and } \nu_M(x_n, x_{n+1}) < \alpha_n.$$

Similarly, $x_m \in \mathcal{B}(x_{m-1}, \alpha_{m-1})$ implies that

$$\mu_M(x_{m-1}, x_m) > 1 - \alpha_{m-1},\ \sigma_M(x_{m-1}, x_m) > 1 - \alpha_{m-1} \text{ and } \nu_M(x_{m-1}, x_m) < \alpha_{m-1}.$$

Let $\alpha \in]0, 1[$ such that

$$(1 - \alpha_n) * (1 - \alpha_{n+1}) * ... * (1 - \alpha_{m-1}) > 1 - \alpha\,,$$
$$(1 - \alpha_n) \triangleleft (1 - \alpha_{n+1}) \triangleleft ... \triangleleft (1 - \alpha_{m-1}) > 1 - \alpha$$
$$\text{and } \alpha_n \diamond \alpha_{n+1} \diamond ... \diamond \alpha_{m-1} < \alpha\,.$$

Therefore,

$$\begin{aligned} \mu_M(x_n, x_m) &\geq \mu_M(x_n, x_{n+1}) * \mu_M(x_{n+1}, x_{n+2}) * ... * \mu_M(x_{m-1}, x_m) \\ &> (1 - \alpha_n) * (1 - \alpha_{n+1}) * ... * (1 - \alpha_{m-1}) \\ &> 1 - \alpha\,, \end{aligned}$$

Applying a similar reasoning, we find

$$\begin{aligned} \sigma_M(x_n, x_m) &\geq \sigma_M(x_n, x_{n+1}) \triangleleft \sigma_M(x_{n+1}, x_{n+2}) \triangleleft ... \triangleleft \sigma_M(x_{m-1}, x_m) \\ &> (1 - \alpha_n) \triangleleft (1 - \alpha_{n+1}) \triangleleft ... \triangleleft (1 - \alpha_{m-1}) \\ &> 1 - \alpha \text{ and} \end{aligned}$$

$$\begin{aligned} \nu_M(x_n, x_m) &\leq \nu_M(x_n, x_{n+1}) \diamond \nu_M(x_{n+1}, x_{n+2}) \diamond ... \diamond \nu_M(x_{m-1}, x_m) \\ &< \alpha_n \diamond \alpha_{n+1} \diamond ... \diamond \alpha_{m-1} \\ &< \alpha\,. \end{aligned}$$

□

Hence, (x_n) is a Cauchy sequence. Since X is complete, then (x_n) converges to y in X. As $y \in X$, there exists $i_0 \in I$ such that $y \in U_{i_0}$. As U_{i_0} is open, then it contains $\mathcal{B}(y, \beta)$ where $\beta \in]0, 1[$, and hence, for n so large we have

$$\mu_M(x_n, y) > 1 - \beta,\ \sigma_M(x_n, y) > 1 - \beta \text{ and } \nu_M(x_n, y) < \beta \text{ with } 1 - \alpha_n > 1 - \beta \text{ and } \alpha_n < \beta\,.$$

Let $x \in \mathcal{B}(x_n, \alpha_n)$. It holds that

$$\mu_M(x, x_n) > 1 - \alpha_n,\ \sigma_M(x, x_n) > 1 - \alpha_n \text{ and } \nu_M(x, x_n) < \alpha_n \text{ for any } x \in X\,.$$

Thus,

$$\mu_M(x, y) \geq \mu_M(x, x_n) * \mu_M(x_n, y) \geq (1 - \beta) * (1 - \beta) > 1 - r\,,$$

$$\sigma_M(x, y) \geq \sigma_M(x, x_n) \triangleleft \sigma_M(x_n, y) \geq (1 - \beta) \triangleleft (1 - \beta) > 1 - r \text{ and}$$

$$\nu_M(x, y) \leq \nu_M(x, x_n) \diamond \nu_M(x_n, y) < \beta \diamond \beta < r\,.$$

This implies that $x \in \mathcal{B}(y, r)$, and hence $\mathcal{B}(x_n, \alpha_n) \subseteq \mathcal{B}(y, r)$. This means that $\mathcal{B}(x_n, \alpha_n)$ admits U_{i_0} as a finite subcover. This is a contradiction. Hence X is compact.

Theorem 4.10. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. Then it holds that X is compact if and only if X is totally bounded and complete.*

Proof. Suppose that X is compact. From Proposition 4.6, it then follows that X is totally bounded. Moreover, Proposition 4.8 then guarantees that X is complete. Thus, X is totally bounded and complete. The converse implication, follows immediately from Proposition 4.9. □

5 Conclusion

In this paper, we have introduced the notions of complete standard single valued neutrosophic metric space and compact standard single valued neutrosophic metric space and we have investigated their most interesting properties and characterizations. In a future work, we plan to study other topological properties for standard single valued neutrosophic metric space such as convexity, connexity and density. Moreover, we intend to use these topological properties to study some fixed point theorems.

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