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Construction of Intuitionistic Fuzzy Mappings with Applications

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Abstract

In a recent paper, Ismail and Massa'deh have introduced the notion of L-fuzzy mapping and some basic operations were proved. In this paper, we generalize this notion to the setting of intuitionistic fuzzy sets. Moreover, we study the main properties related to intuitionistic fuzzy mapping. As applications, we provide properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological spaces and investigate the relation among various kinds of intuitionistic fuzzy continuity.

1. Introduction

Mappings in crisp set theory are very well known and play a prominent role in mathematical branches such as topology and its analysis approaches. They appear to enhance the concept of functional predicate in formal logic [14] and also closely related to category theory [23]. In dynamical systems, a mapping denotes an evolution function used to create discrete dynamical systems [11].

In fuzzy setting, several authors introduce and investigate the concept of fuzzy mapping in different ways. Heilpern [13] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings. Ismail and Massa'deh [10] defined L-fuzzy mappings and studied their operations, also they developed many properties of classical mappings into L-fuzzy case. Lim et al. [19] investigated the equivalence relations and mappings for fuzzy sets and relationship among them.

In 1983, Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of Zadeh's fuzzy set previously introduced in [24] by using two membership functions for the elements of the universe of discourse. After that, several intuitionistic fuzzy concepts are studied by many authors. For the concept of mapping, an extended approaches are proposed based on Atanassov's intuitionistic fuzzy sets. Kang et al. [18] introduced the concept of intuitionistic fuzzy mapping and they give the decomposition of an intuitionistic fuzzy mapping by using intuitionistic fuzzy equivalence relations. Shen et al. [22] presented the notion of intuitionistic fuzzy mapping as a generalization of fuzzy mapping, and they established the decomposition and representation theorems of intuitionistic fuzzy mappings. Very recently, Gomathi and Jayanthi [12] introduced the concept of intuitionistic fuzzy $b^\sharp$ continuous mapping in intuitionistic fuzzy topological spaces and discussed some of their properties and characterizations. For more details about intuitionistic fuzzy mappings and background, the readers are referred to [16, 20, 25] and more others.

In this paper, we continue further by generalizing the notion of fuzzy mapping introduced by Ismail and Massa'deh to the intuitionistic fuzzy setting. Hereafter, the main properties related to intuitionistic fuzzy mapping are studied. Also, we generalize the notion of fuzzy topology on fuzzy sets to the intuitionistic fuzzy case to provide properties of intuitionistic fuzzy continuous mappings. To that end, the relations among intuitionistic fuzzy continuity, precontinuity and α-continuity are investigated.

This paper is structured as follows. After recalling some basic definitions and properties in Section 2, the notion of intuitionistic fuzzy mapping by construction on a set is introduced, and some basic properties are given in Section 3. As applications, some properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological space are provided and relations among some kinds of intuitionistic fuzzy continuity in Section 4 are investigated. Finally, some conclusions and future research in Section 5 are presented.

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2. Preliminaries

This section contains the basic definitions and properties of intuitionistic fuzzy sets, intuitionistic fuzzy relations and some related notions that will be needed throughout this paper.

2.1. Atanassov's intuitionistic fuzzy sets

In this subsection we recall some basic concepts of intuitionistic fuzzy sets.

Let X be a universe, then a fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ defined by Zadeh [24] is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of a membership of the element x in the fuzzy subset A for each $x \in X$.

Atanassov in [1] introduced another fuzzy object, called intuitionistic fuzzy set as a generalization of the concept of fuzzy set, shown as follows

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$, with the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1\,, \tag{2.1}$$

for any $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy set A for each $x \in X$.

In the fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard negation) and thus it is fixed. In intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \le 1 - \mu_A(x)$. Certainly fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$, but not conversely.

Throughout this paper, authors denote the set of all intuitionistic fuzzy sets in a set X as $IFS(X)$ and $X, Y, Z, \ldots$ etc., will be nonempty crisp sets.

Definition 2.1. [1] *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$, be two IFSs on a set X. Then*

(i) $A \subseteq B$ *if* $\mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$, *for all* $x \in X$,
(ii) $A = B$ *if* $\mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$, *for all* $x \in X$,
(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,
(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,
(v) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$,
(vi) $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$,
(vii) $\langle A\rangle = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

For more details please refer to ([1–3, 21, 25]).

Definition 2.2. [3] *Let A be an intuitionistic fuzzy set on universe X. The support of A is the crisp subset of X given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

In the sequel, we need the following definition of level set (which is also often called (α, β)-cut) of intuitionistic fuzzy set.

Definition 2.3. [15] *Let A be an intuitionistic fuzzy set on a nonempty set X. The (α, β)-cut of A is the crisp subset*

$$A_{(\alpha,\beta)} = \{x \in X \mid \mu_A(x) \ge \alpha \text{ and } \nu_A(x) \le \beta\},$$

where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$.

2.2. Intuitionistic fuzzy relations

Burillo and Bustince [4, 5] introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 2.4. [4, 5] *An intuitionistic fuzzy binary relation (An intuitionistic fuzzy relation, for short) from a universe X to a universe Y is an intuitionistic fuzzy subset in $X \times Y$, i.e., is an expression R given by*

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid (x,y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \to [0,1], \text{ and } \nu_A : X \times Y \to [0,1]$$

satisfy the condition

$$0 \le \mu_R(x,y) + \nu_R(x,y) \le 1\,, \tag{2.2}$$

for any $(x,y) \in X \times Y$. The value $\mu_R(x,y)$ is called the degree of a membership of (x,y) in R and $\nu_R(x,y)$ is called the degree of a non-membership of (x,y) in R.

Next, the following definitions is needed to recall.

Definition 2.5. *Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y.*

(i) The transpose (inverse) R^t of R is the intuitionistic fuzzy relation from the universe Y to the universe X defined by

$$R^t = \{\langle (x,y), \mu_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\},$$

where

$$\begin{cases} \mu_{R^t}(x,y) = \mu_R(y,x) \\ \quad \text{and} \\ \nu_{R^t}(x,y) = \nu_R(y,x), \end{cases}$$

for any $(x,y) \in X \times Y$.

(ii) R is said to be contained in P or we say that P contains R, denoted by $R \subseteq P$, if for all $(x,y) \in X \times Y$ it holds that $\mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.

(iii) The intersection (resp. the union*) of two intuitionistic fuzzy relations R and P from a universe X to a universe Y is an intuitionistic fuzzy relation defined as*

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}$$

and

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}.$$

The following properties are crucial in this paper (see e.g. [4, 5, 8]).

Definition 2.6. *Let R be an intuitionistic fuzzy relation from a universe X into itself.*

(i) Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.*

(ii) Antisymmetry*: for any $x,y \in X$, $x \neq y$ then*

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases},$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) Perfect antisymmetry*: for any $x,y \in X$ with $x \neq y$ and*

$$\begin{cases} \mu_R(x,y) > 0 \\ \quad \text{or} \\ \mu_R(x,y) = 0 \ \text{ and } \nu_R(x,y) < 1, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ \quad \text{and} \\ \nu_R(y,x) = 1. \end{cases}$$

(iv) Transitivity*: $R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R$.*

In the above definition, the composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ used in the transitivity means that

$$R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\},\ \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\}\rangle \mid x,z \in X\},$$

where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\} \leq 1,$$

for any $x,z \in X$.

The properties of this composition and the choice of α, β, λ and ρ, for which this composition fulfills a maximal number of properties, are investigated in [4]- [8].

3. Construction of intuitionistics fuzzy mappings

In crisp set theory, mappings are defined as binary relations. In this section, the notion of intuitionistic fuzzy mapping as intuitionistic fuzzy relations by construction on a set is introduced, and some basic properties are given.

Definition 3.1. *Let A be an intuitionistic fuzzy set on X and B be an intuitionistic fuzzy set on Y, let $f : Supp\,A \to Supp\,B$ be an ordinary mapping and R be an intuitionistic fuzzy relation on $X \times Y$. Then f_R is called an intuitionistic fuzzy mapping if for all $(x,y) \in Supp\,A \times Supp\,B$ the following condition is satisfied:*

$$\mu_R(x,y) = \begin{cases} min(\mu_A(x), \mu_B(f(x)) \ , \ if\ y = f(x) \\ 0\ , \ Otherwise\, , \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} max(\nu_A(x), \nu_B(f(x)) \ , \ if\ y = f(x) \\ 1\ , \ Otherwise\, , \end{cases}$$

with $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$

Example 3.2. *Let $X = \{\alpha, \beta\}$, $Y = \{1,2,3\}$, $A \in IFS(X)$ and $B \in IFS(Y)$ given by :*

$$A = \{\langle \alpha, 0.5, 0.2\rangle, \langle \beta, 0.1, 0.7\rangle\} \text{ and } B = \{\langle 1,0,1\rangle, \langle 2, 0.1, 0.5\rangle, \langle 3, 0.7, 0.2\rangle\}$$

.

We will construct the intuitionistic fuzzy mapping f_R by :

(i) an ordinary mapping $f : \{\alpha, \beta\} \to \{2,3\}$ such that $f(\alpha) = 2$ and $f(\beta) = 3$,

(ii) an intuitionistic fuzzy relation R defined by :
$\mu_R(\alpha, f(\alpha)) = \mu_R(\alpha, 2) = \mu_A(\alpha) \wedge \mu_B(2) = 0.1$
$\mu_R(\beta, f(\beta)) = \mu_R(\beta, 3) = \mu_A(\beta) \wedge \mu_B(3) = 0.1$
$\mu_R(\alpha, 1) = \mu_R(\alpha, 3) = \mu_R(\beta, 1) = \mu_R(\beta, 2) = 0$
In similar way, it holds that
$\nu_R(\alpha, f(\alpha)) = \nu_R(\alpha, 2) = \nu_A(\alpha) \vee \nu_B(2) = 0.5$
$\nu_R(\beta, f(\beta)) = \nu_R(\beta, 3) = \nu_A(\beta) \vee \nu_B(3) = 0.7$
$\nu_R(\alpha, 1) = \nu_R(\alpha, 3) = \nu_{R_I}(\beta, 1) = \nu_{R_I}(\beta, 2) = 1.$
Hence, $\mu_R(x,y) = \{\langle(\alpha, f(\alpha)), 0.1, 0.5\rangle, \langle(\beta, f(\beta)), 0.1, 0.7\rangle, \langle(\alpha,1),0,1\rangle,$
$\langle(\alpha,3),0,1\rangle, \langle(\beta,1),0,1\rangle, \langle(\beta,2),0,1\rangle\}.$

Thus, f_R is an intuitionistic fuzzy mapping.

Remark 3.3. *From the above definition, we can construct the intuitionistic fuzzy mapping by this method*

(i) We determine the Supp A and Supp B.

(ii) We determine the ordinary mapping from Supp A to Supp B.

(iii) We determine the intuitionistic fuzzy relation R to get the relationship degree and non-relationship degree between each element and its image.

(iv) Finally, we conclude the construction of the intuitionistic fuzzy mapping.

Definition 3.4. *Let f_R, g_S be two intuitionistic fuzzy mappings, then f_R and g_S are equal if and only if $f = g$ and $R = S$ i.e., $(\mu_R(x, f(x)) = \mu_S(x, g(x))$ and $\nu_R(x, f(x)) = \nu_S(x, g(x)))$.*

Definition 3.5. *Let A be an intuitionistic fuzzy set on X, let $f : Supp\,A \to Supp\,A$ be an ordinary mapping such that $f(x) = x$ and R be an intuitionistic fuzzy relation on $X \times X$. Then f_R is called an intuitionistic fuzzy identity mapping if for all $x,y \in Supp\,A$ the following condition is satisfied:*

$$\mu_R(x,y) = \begin{cases} \mu_A(x)\ , \ if\ x = y \\ 0\ , \ Otherwise\, , \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} \nu_A(x)\ , \ if\ x = y \\ 1\ , \ Otherwise\, , \end{cases}$$

with $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1.$

Definition 3.6. *Let A, B and C are an intuitionistic fuzzy sets on X, Y and Z respectively, let $f : Supp\,A \to Supp\,B$ and $g : Supp\,B \to Supp\,C$ are an ordinary mappings and R, S are an intuitionistic fuzzy relations on $X \times Y$ and $Y \times Z$ respectively. Then $(g \circ f)_T$ is called the composition of intuitionistic fuzzy mappings f_R and g_R such that $g \circ f : Supp\,A \to Supp\,C$ and the intuitionistic fuzzy relation T is defined by*

$$\begin{cases} \mu_T(x,z) = sup_y(min(\mu_R(x,y), \mu_S(y,z))) \\ \text{and} \\ \nu_T(x,z) = inf_y(max(\nu_R(x,y), \nu_S(y,z)))\, , \end{cases}$$

for any $(x,z) \in Supp\,A \times Supp\,C$.

Remark 3.7. *The intuitionistic fuzzy relation T in the above definition can be written as follows:*
$\mu_T(x,z) = min\{\mu_A(x),\ \mu_B(f(x)), \mu_C(g(f(x))\}$ *and* $\nu_T(x,z) = max\{\nu_A(x),\ \nu_B(f(x)), \nu_C(g(f(x))\}$.
Indeed, for any $(x,z) \in Supp\ A \times Supp\ C$, *we have*

$$\begin{aligned}
\mu_T(x,z) &= \mu_T(x, g(f(x))) \\
&= \mu_{S\circ R}(x, g(f(x))) \\
&= sup_y\{min\{\mu_R(x,y),\ \mu_S(y, g(f(x))) \\
&= min\{\mu_R(x, f(x))),\ \mu_S(f(x)), g(f(x)))\} \\
&= min\{\mu_A(x),\ \mu_B(f(x)), \mu_C(g(f(x))\}.
\end{aligned}$$

Similarly, for any $(x,z) \in Supp\ A \times Supp\ C$, *it holds that*

$$\begin{aligned}
\nu_T(x,z) &= \nu_T(x, g(f(x))) \\
&= \nu_{S\circ R}(x, g(f(x))) \\
&= inf_y\{max\{\nu_R(x,y),\ \nu_S(y, g(f(x))) \\
&= max\{\nu_R(x, f(x))),\ \nu_S(f(x)), g(f(x)))\} \\
&= max\{\nu_A(x),\ \nu_B(f(x)), \nu_C(g(f(x))\}.
\end{aligned}$$

Example 3.8. *Let* $X = \mathbb{N}$, $Y = \mathbb{R}$ *and* $Z = \mathbb{R}$, *and let* $A \in IFS(X)$, $B \in IFS(Y)$ *and* $C \in IFS(Z)$, *defined as follows :*

$$\mu_A(n) = \frac{1}{1+n} \text{ and } \nu_A(n) = \tfrac{n}{2+2n}, \text{ for any } n \in \mathbb{N}$$

.

$$\mu_B(x) = \begin{cases} 0.25\ ,\ if\ x \in [-1,1] \\ 0\ ,\ Otherwise, \end{cases} \quad and \quad \nu_B(x) = \begin{cases} 0.5\ ,\ if\ x \in [-1,1] \\ 1\ ,\ Otherwise, \end{cases}$$

$$\mu_C(x) = \frac{|\ cos(x)\ |}{3} \text{ and } \nu_C(x) = \tfrac{|sin(x)|}{3}$$

for any $x \in \mathbb{R}$.
We define an intuitionistic fuzzy mappings $f_R : A \to B$ *and* $g_S : B \to C$ *by :*

(i) an ordinary mappings $f : Supp\ A \longrightarrow Supp\ B$, *defined for any* $n \in Supp\ A$ *by :*

$$f(n) = \begin{cases} 1\ ,\ if\ n \text{ is even number}, \\ -1\ , if\ n \text{ odd is number}, \end{cases}$$

and $g : Supp\ B \longrightarrow Supp\ C$ *defined by* $g(x) = 2x$, *for any* $x \in [-1,1]$.

(ii) an IF-relations R *and* S *defined by :*

$\mu_R(n, f(n)) = \wedge\{\mu_A(n), \mu_B(f(n))\} = \wedge\{\frac{1}{1+n}, 0.25\}$ *and* $\nu_R(n, f(n)) = \vee\{\nu_A(n), \nu_B(f(n))\} = \vee\{\frac{n}{2+2n}, 0.5\}$ *and* $\mu_S(x, g(x)) = \wedge\{\mu_B(x), \mu_C(g(x))\} = \begin{cases} \wedge\{0.25,\ \frac{|cos(2x)|}{3}\}\ ,\ x \in [-1,1], \\ 0\ ,\ otherwise, \end{cases}$
and $\nu_S(x, g(x)) = \vee\{\nu_B(x), \nu_C(g(x))\} = \begin{cases} \vee\{0.5,\ \frac{|sin(2x)|}{3}\}\ ,\ x \in [-1,1], \\ 1\ ,\ otherwise, \end{cases}$

Then, the composition $g_S \circ f_R = (g \circ f)_T$ *is defined by :*

(i) an ordinary mapping $f : Supp\ A \longrightarrow Supp\ C$, *defined for any* $n \in Supp\ A$ *by :*

$$(g\circ f)(n) = \begin{cases} 2\ ,\ if\ n \text{ is even number}, \\ -2\ ,\ if\ n \text{ is odd number}, \end{cases}$$

(ii) an IF-relation T *defined by :*

$$\begin{aligned}
\mu_T(n, (g\circ f)(n)) &= \begin{cases} \wedge\{\frac{1}{1+n}, 0.25,\ \frac{|cos(2)|}{3}\}\ ,\ if\ n \text{ is even number} \\ \wedge\{\frac{1}{1+n}, 0.25,\ \frac{|cos(-2)|}{3}\}\ ,\ if\ n \text{ is odd number} \end{cases} \\
&= \wedge\{\frac{1}{1+n}, 0.25,\ \frac{|\ cos(2)\ |}{3}\} \\
&= \wedge\{\frac{1}{1+n}, 0.25\},
\end{aligned}$$

$$\nu_T(n,(g\circ f)(n)) = \begin{cases} \vee\{\frac{n}{2+2n}, 0.25, \frac{|sin(2)|}{3}\}, & if\ n\, is\ even\ number \\ \vee\{\frac{n}{2+2n}, 0.25, \frac{|sin(-2)|}{3}\}, & if\ n\, is\ odd\ number \end{cases}$$
$$= \vee\{\frac{n}{2+2n}, 0.25, \frac{|\ sin(2)\ |}{3}\}$$
$$= \vee\{\frac{2}{2+2n}, 0.25\}.$$

Proposition 3.9. *The composition of intuitionistic fuzzy mappings is an associative operation.*

Proof. Let A,B,C and D are an intuitionistic fuzzy sets on X,Y,Z and T respectively, let $f_{R_1}: A\to B$, $g_{R_2}: B\to C$ and $h_{R_3}: C\to D$ are an intuitionistic fuzzy mappings. We need to show that $h_{R_3}\circ(g_{R_2}\circ f_{R_1}) = (h_{R_3}\circ g_{R_2})\circ f_{R_1}$. On the one hand, it is easy to verify that $(h\circ(g\circ f)) = ((h\circ g)\circ f)$. On the one hand,

$$\begin{aligned}
\mu_{R_3\circ(R_2\circ R_1)}(x,h\circ(g\circ f)(x)) &= min\{\mu_{R_2\circ R_1}(x,g\circ f(x)),\ \mu_{R_3}(g\circ f(x)),h\circ(g\circ f)(x)\}\} \\
&= min\{min\{\mu_{R_1}(x,f(x)),\ \mu_{R_2}(f(x),g(f(x))\},\ \mu_{R_3}(g\circ f(x),h\circ(g\circ f)(x))\} \\
&= min\{\mu_{R_1}(x,f(x)),\ \mu_{R_2}(f(x),g(f(x)),\ \mu_{R_3}(g\circ f(x),h\circ(g\circ f)(x))\} \\
&= min\{\mu_A(x),\ \mu_B(f(x)),\ \mu_C(g(f(x)),\ \mu_D((h\circ g)\circ f)(x))\} \\
&= min\{\mu_{R_1}(x,f(x)),\ \mu_{R_2}(f(x),g(f(x)),\ \mu_{R_3}(g\circ f(x),(h\circ g)\circ f)(x))\} \\
&= min\{\mu_{R_1}(x,f(x)),\ min\{\mu_{R_2}(f(x),g(f(x))),\ \mu_{R_3}(g\circ f(x),(h\circ(g\circ f))(x))\}\} \\
&= min\{\mu_{R_1}(x,f(x)),\ \mu_{R_3\circ R_2}(f(x),(g\circ f)(x))\}\} \\
&= \mu_{(R_3\circ R_2)\circ R_1}(x,((h\circ g)\circ f)(x))
\end{aligned}$$

In similar way, we prove that $\nu_{R_3\circ(R_2\circ R_1)}(x,h\circ(g\circ f)(x)) = \nu_{(R_3\circ R_2)\circ R_1}(x,(h\circ g)\circ f(x))$.

□

Remark 3.10. *The intuitionistic fuzzy identity mapping Id_R is neutral for the composition of intuitionistic fuzzy mappings.*

In the sequel, we need to introduce the notion of the direct image and the inverse image of intuitionistic fuzzy set by an intuitionistic fuzzy mapping.

Definition 3.11. *Let $f_R: A\to B$ be an intuitionistic fuzzy mapping from an intuitionistic fuzzy set A to another intuitionistic fuzzy set B and $C\subseteq A$. The direct image of C by f_R is defined by $f_R(C) = \{\langle y,\mu_{f_R(C)}(y),\nu_{f_R(C)}(y)\rangle \mid y\in Y\}$, where*

$$\mu_{f_R(C)}(y) = \begin{cases} \mu_B(y), & if\ y\in f(supp(C)) \\ 0, & Otherwise, \end{cases}$$

and

$$\nu_{f_R(C)}(y) = \begin{cases} \nu_B(y), & if\ y\in f(supp(C)) \\ 1, & Otherwise, \end{cases}$$

Similarly, if $C'\subseteq B$. The inverse image of C' by f is defined by $f_R^{-1}(C') = \{\langle x,\mu_{f_R^{-1}(C')}(x),\nu_{f_R^{-1}(C')}(x)\rangle \mid x\in X\}$, where

$$\mu_{f_R^{-1}(C')}(x) = \begin{cases} \mu_A(x), & if\ x\in f^{-1}(supp(C')) \\ 0, & Otherwise, \end{cases}$$

and

$$\nu_{f_R^{-1}(C')}(x) = \begin{cases} \nu_A(x), & if\ x\in f^{-1}(supp(C')) \\ 1, & Otherwise, \end{cases}$$

Example 3.12. *Let $X=\mathscr{P}(\mathbb{R})$, $Y=\{\alpha,\beta\}$ and $A\in IFS(X)$ defined for any $S\in\mathscr{P}(\mathbb{R})$ by :*

$$\mu_A(S) = \begin{cases} 0.55, & if\ S\, is\ denumerable\ set \\ 0, & Otherwise, \end{cases}$$

$$\nu_A(S) = \begin{cases} 0.3, & if\ S\, is\ denumerable\ set \\ 1, & Otherwise. \end{cases}$$

Also, let $B\in IFS(Y)$ given by $B=\{\langle\alpha,0.2,0.5\rangle,\langle\beta,0.7,0.3\rangle\}$.
We define the intuitionistic fuzzy mapping $f_R: A\to B$ by:

(i) an ordinary mapping $f: Supp\ A\longrightarrow Supp\ B$, defined for any $S\in Supp\ A$ by
$$f(S) = \begin{cases} \alpha, & if\ S\, is\ finite\ set \\ \beta, & Otherwise, \end{cases}$$
(ii) an IF-relation R defined by $\mu_R(S,f(S)) = \mu_A(S)\wedge\mu_B(f(S)) = 0.55\wedge 0.2 = 0.2$ and $\nu_R(S,f(S)) = \nu_A(S)\vee\nu_B(f(S)) = 0.3\vee 0.5 = 0.5$

Now, if we take C an IF-set on X, where $C \subseteq A$ given by:

$$\mu_C(S) = \begin{cases} 0.4\ ,\ if\ S is\ finite\ set \\ 0\ ,\ Otherwise\ , \end{cases}$$

$$\nu_C(S) = \begin{cases} 0.4\ ,\ if\ S is\ finite\ set \\ 1\ ,\ Otherwise. \end{cases}$$

Then, the direct image of C by f_R is defined by :

$$\mu_{f_R(C)}(y) = \begin{cases} \mu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} 0.2\ ,\ if\ y = \alpha \\ 0\ ,\ ,\ if\ y = \beta \end{cases}$$

and

$$\nu_{f_R(C)}(y) = \begin{cases} \mu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} 0.5\ ,\ if\ y = \alpha \\ 1\ ,\ ,\ if\ y = \beta. \end{cases}$$

Moreover, it is easy to show that $f_R(C) \subseteq B$.

Next, if we take C' an IF-set on Y, where $C' \subseteq B$ given by:

$$\mu_{C'}(y) = \begin{cases} 0.1\ ,\ if\ y = \alpha \\ 0\ ,\ y = \beta\ , \end{cases} \quad and \quad \nu_{C'}(y) = \begin{cases} 0.6\ ,\ if\ y = \alpha \\ 1\ ,\ y = \beta. \end{cases}$$

Then, the inverse image of C' by f is defined by :

$$\mu_{f_R^{-1}(C')}(S) = \begin{cases} \mu_A(S)\ ,\ if\ S \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} 0.55\ ,\ if\ S\ is\ finite\ set \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\nu_{f_R^{-1}(C')}(S) = \begin{cases} \nu_A(S)\ ,\ if\ S \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} 0.3\ ,\ if\ S\ is\ finite\ set \\ 1\ ,\ Otherwise. \end{cases}$$

Moreover, it is easy to show that $f_R^{-1}(C') \subsetneq A$ in the case of $S = \mathbb{N}$.

Definition 3.13. *Let A be an intuitionistic fuzzy set on a set X and $\sim$ be an equivalence relation over $Supp(A)$, let B an intuitionistic fuzzy set on $\mathscr{P}(X)$ defined by :*

$$\mu_B(\theta) = \begin{cases} \mu_A(x)\ ,\ if \theta = \bar{x} \in supp(A)/\sim \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\nu_B(\theta) = \begin{cases} \nu_A(x)\ ,\ if \theta = \bar{x} \in supp(A)/\sim \\ 1\ ,\ Otherwise. \end{cases}$$

Then, the intuitionistic fuzzy mapping $P_R : A \longrightarrow B$ defined by :

(i) an ordinary mapping $P : Supp(A) \longrightarrow Supp(B)$ such that $P(x) = \bar{x}$ for any $x \in Supp(A)$,
(ii) an intuitionistic fuzzy relation R defined by :

$$\begin{aligned} \mu_R(x, P(x)) &= min\{\mu_A(x),\ \mu_B(P(x))\} \\ &= min\{\mu_A(x),\ \mu_B(\bar{x})\} \\ &= min\{\mu_A(x),\ \mu_A(x)\} \\ &= \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \nu_R(x, P(x)) &= max\{\nu_A(x),\ \nu_B(P(x))\} \\ &= max\{\nu_A(x),\ \nu_B(\bar{x})\} \\ &= max\{\nu_A(x),\ \nu_A(x)\} \\ &= \nu_A(x) \end{aligned}$$

is called the intuitionistic fuzzy projection mapping.

Now, we define the product of intuitionistic fuzzy sets and intuitionistic fuzzy projection mappings.

Definition 3.14. *Let A be an intuitionistic fuzzy set on X and B be an intuitionistic fuzzy set on Y. The product of A and B, denoted by $A \times B$ is an intuitionistic fuzzy set on $X \times Y$ defined by :*
$\mu_{X\times Y}(x,y) = min\{\mu_A(x), \mu_B(y)\}$ *and* $\nu_{X\times Y}(x,y) = max\{\nu_A(x), \nu_B(y)\}$.
Also, we define the first intuitionistic fuzzy projection mapping $(P_1)_R : A \times B \longrightarrow A$ by:

(i) an ordinary mapping $P_1 : Supp(A \times B) \longrightarrow Supp(A)$ such that $P_1(x,y) = x$ for any $(x,y) \in Supp(A \times B)$,

(i) an intuitionistic fuzzy relation R defined by :

$$\begin{aligned} \mu_R((x,y),P_1(x,y)) &= min\{\mu_{A\times B}(x,y),\ \mu_A(P_1(x,y))\} \\ &= min\{\mu_A(x),\ \mu_B(y),\ \mu_A(x)\} \\ &= min\{\mu_A(x),\ \mu_B(y)\} \end{aligned}$$

and

$$\begin{aligned} \nu_R((x,y),P_1(x,y)) &= max\{\nu_{A\times B}(x,y),\ \nu_A(P_1(x,y))\} \\ &= max\{\nu_A(x),\ \nu_B(y),\ \nu_A(x)\} \\ &= max\{\nu_A(x),\ \nu_B(y)\} \end{aligned}$$

The second intuitionistic fuzzy projection mapping is defined analogously.

Next, we introduce the notion of disjoint union of intuitionistic fuzzy sets and intuitionistic fuzzy inclusion mappings.

Definition 3.15. *Let A be an intuitionistic fuzzy set on X and B be an intuitionistic fuzzy set on Y. The disjoint union of A and B, denoted by $A\sqcup B$ is an intuitionistic fuzzy set on $X\times\{1\}\cup Y\times\{2\}$ defined by :*

$$\mu_{A\sqcup B}(x,k) = \begin{cases} \mu_A(x)\ ,\ if\, k=1 \\ \mu_B(x)\ ,\ if\, k=2 \end{cases}$$

and

$$\nu_{A\sqcup B}(x,k) = \begin{cases} \nu_A(x)\ ,\ if\, k=1 \\ \nu_B(x)\ ,\ if\, k=2 \end{cases}$$

Also, we define the first intuitionistic fuzzy inclusion mapping $(\varphi_1)_R : A \longrightarrow A\sqcup B$ by :

(i) an ordinary mapping φ_1, defined by :

$$\varphi_1 : Supp(A) \longrightarrow Supp(A\sqcup B) \text{ such that } \varphi_1(x) = (x,1) \text{ for any } x\in Supp(A),$$

(ii) an intuitionistic fuzzy relation R defined by :

$$\begin{aligned} \mu_R(x,\varphi_1(x)) &= min\{\mu_A(x),\ \mu_{A\sqcup B}(\varphi_1(x))\} \\ &= min\{\mu_A(x),\ \mu_{A\sqcup B}(x,1)\} \\ &= min\{\mu_A(x),\ \mu_A(x)\} \\ &= \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \nu_R(x,\varphi_1(x)) &= max\{\nu_A(x),\ \nu_{A\sqcup B}(\varphi_1(x))\} \\ &= max\{\nu_A(x),\ \nu_{A\sqcup B}(x,1)\} \\ &= max\{\nu_A(x),\ \nu_A(x)\} \\ &= \nu_A(x) \end{aligned}$$

The second intuitionistic fuzzy inclusion mapping is defined analogously.

4. Applications

In this section, we establish as an application the intuitionistic fuzzy continuous mapping in intuitionistic fuzzy topological spaces.

4.1. Intuitionistic fuzzy topology

This subsection is devoted to study the structure of intuitionistic fuzzy topology as a generalization of the structure of fuzzy topology given by Kandil et al. [17].

Definition 4.1. *Let A be an intuitionistic fuzzy set on the set X and $O_A = \{U \text{ is an IFS on } X :\ U\subseteq A\}$. We define an intuitionistic fuzzy topology on intuitionistic fuzzy set A by the family $T\subseteq O_A$ which satisfies the following conditions :*

(i) $A,\ 0_\sim \in T$;
(ii) if $U_1, U_2\in T$, then $U_1\cap U_2\in T$;
(iii) if $U_i\in T$ for all $i\in I$, then $\cup_I U_i\in T$.

T is called an intuitionistic fuzzy topology of A and the pair (A,T) is an intuitionistic fuzzy topological space (IF-TOP, for short). Every element of T is called an intuitionistic fuzzy open set (IFOS, for short).

Example 4.2. *Let X be a nonempty set and A be an intuitionistic fuzzy set on $\mathscr{P}(X)$ given by :* $\mu_A(\theta) = \begin{cases} 1\ , \ if\ \theta = \emptyset \\ 0.5\ , \ 0 < |\theta| < \infty, \\ 0\ , \ Otherwise, \end{cases}$

and $\nu_A(\theta) = \begin{cases} 0\ , \ if\ \theta = \emptyset \\ 0.4\ , \ 0 < |\theta| < \infty, \\ 0.2\ , \ Otherwise, \end{cases}$

Then, the family $T = \{A, 0_\sim, U\}$ where:

$\mu_U(\theta) = \begin{cases} 0.4\ , \ |\theta| < \infty, \\ 0\ , \ Otherwise, \end{cases}$ *and* $\nu_U(\theta) = \begin{cases} 0.6\ , \ |\theta| < \infty, \\ 0.5\ , \ Otherwise, \end{cases}$

is an intuitioniste fuzzy topology on A.

Inspired by the notion of interior (resp. closure) on intuitionistic fuzzy topological space on a set introduced by Atanassov [3], authors define these notions in intuitionistic fuzzy topology on an intuitionistic fuzzy set.

Definition 4.3. *Let (A,T) be an intuitionistic fuzzy topological space, for every intuitionistic fuzzy subset G of X we define the interior and closure of G by:*

$$int(G) = \{\langle x, \max_{x\in X} \mu_U(x), \min_{x\in X} \nu_U(x)\rangle \mid x \in U \subseteq G\}$$

and

$$cl(G) = \{\langle x, \min_{x\in X} \mu_K(x), \max_{x\in X} \nu_K(x)\rangle \mid x \in A \text{ and } G \subseteq K\}$$

Example 4.4. *Let $X = \{a,b,c\}$ and $A,B,C,D \in IFS(X)$ such that*
$A = \{< a, 0.5, 0.1 >, < b, 0.7, 0.2 >, < c, 0.6, 0 >\}$
$B = \{< a, 0.5, 0.2 >, < b, 0.5, 0.4 >, < c, 0.4, 0.4 >\}$
$C = \{< a, 0.4, 0.5 >, < b, 0.6, 0.3 >, < c, 0.2, 0.3 >\}$
$D = \{< a, 0.5, 0.2 >, < b, 0.6, 0.3 >, < c, 0.4, 0.3 >\}$
$E = \{< a, 0.4, 0.5 >, < b, 0.5, 0.4 >, < c, 0.2, 0.4 >\}$
Then the family $T = \{A, 0_\sim, B, C, D, E\}$ is an IFT of A.
Now, we suppose that $G \in IFS(X)$ given by $G = \{< a, 0.41, 0.49), < b, 0.61, 0.29 >, < c, 0.2, 0.2 >\}$. Then, $int(G) = C \cup E = C$ and $cl(G) = 1_\sim$.

Definition 4.5. [9] *Let (A,T) be an intuitionistic fuzzy topological space and $U \in IFS(A,T)$. Then U is called :*

1. *an intuitionistic fuzzy semiopen set (IFSOS) if $U \subseteq cl(int(U))$;*
2. *an intuitionistic fuzzy α-open set (IFαOS) if $U \subseteq int(cl(int(U)))$;*
3. *an intuitionistic fuzzy preopen set (IFPOS) if $U \subseteq int(cl(U))$;*
4. *an intuitionistic fuzzy regular open set (IFROS) if $U = int(cl(U))$.*

4.2. Intuitionistic fuzzy continuous mappings

The present section contains an interesting properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological space and relations between various kinds of intuitionistic fuzzy continuous mapping. First, the notion of intuitionistic fuzzy continuous mapping is introduced.

Definition 4.6. *Let (A,T) (B,L) be two intuitionistic fuzzy topological spaces. The mapping $f_R : (A,T) \to (B,L)$ is an intuitionistic fuzzy continuous if and only if the inverse of each L-open intuitionistic fuzzy set is T-open intuitionistic fuzzy set.*

Example 4.7. *Let (A,T) and (B,T') be two intuitionistic fuzzy topologies, where*
$\mu_A(x) = 0.55$ and $\nu_A(x) = 0.4$, for any $x \in \mathbb{R}$ and

$$\mu_B(y) = \begin{cases} 0.5\ , \ if\ y \geq 0 \\ 0.8\ , \ Otherwise, \end{cases}$$

and

$$\nu_B(y) = \begin{cases} 0.2\ , \ if\ y \geq 0 \\ 0.1\ , \ Otherwise, \end{cases}$$

We suppose that $T = \{A, 0_\sim, U_1\}$, where

$\mu_{U_1}(x) = \begin{cases} 0.55\ , \ if\ x \in \mathbb{R} \setminus [-2,0] \\ 0\ , \ Otherwise, \end{cases}$ *and* $\nu_{U_1}(x) = \begin{cases} 0.4\ , \ if\ x \in \mathbb{R} \setminus [-2,0] \\ 1\ , \ Otherwise, \end{cases}$

Also, we suppose that $T' = \{B, 0_\sim, U_1'\}$, where

$\mu_{U_1'}(y) = \begin{cases} 0.5\ , \ if\ y \geq 0 \\ 0\ , \ Otherwise, \end{cases}$ *and* $\nu_{U_1'}(y) = \begin{cases} 0.3\ , \ if\ y \geq 0 \\ 1\ , \ Otherwise. \end{cases}$

Then, the intuitionistic fuzzy mapping $f_R : A \to B$ define by :

(i) an ordinary mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = (x+1)^2 - 1$ for any $x \in \mathbb{R}$,

(i) an intuitionistic fuzzy relation R defined by :

$$\mu_R(x,f(x)) = \begin{cases} 0.5\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 0.55\ ,\ Otherwise\,, \end{cases} \qquad and\ \nu_R(x,f(x)) = 0.4.$$

is an intuitionistic fuzzy continuous mapping.
Indeed, it is easy to show that $f_R^{-1}(B) = A$ *and* $f_R^{-1}(0_\sim) = 0_\sim$ *and we have,*

$$\begin{aligned} \mu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \mu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_1')) \\ 0\ ,\ Otherwise\,, \end{cases} \\ &= \begin{cases} \mu_A(x)\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 0\ ,\ Otherwise\,, \end{cases} \\ &= \mu_{U_1}(x), \end{aligned}$$

and

$$\begin{aligned} \nu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \nu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_1')) \\ 1\ ,\ Otherwise\,, \end{cases} \\ &= \begin{cases} \nu_A(x)\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 1\ ,\ Otherwise\,, \end{cases} \\ &= \begin{cases} 0.4\ ,\ if\ x\in\mathbb{R}\setminus[-2,0] \\ 1\ ,\ Otherwise\,, \end{cases} \\ &= \nu_{U_1}(x). \end{aligned}$$

Hence, $f_R^{-1}(U_1') = U_1 \in T$*. Thus,* f_R *is an intuitionistic fuzzy continuous mapping.*

Remark 4.8. *Let* (A,T) *be an intuitionistic fuzzy topological space. Then the intuitionistic fuzzy identity mapping* $Id_R : (A,T) \to (A,T)$ *is an intuitionistic fuzzy continuous mapping.*

Next, the relations between various kinds of intuitionistic fuzzy continuous mapping are provided. First, the definitions of precontinuous mapping, α-continuous mapping introduced by Gürçay et al. [9] need to be recalled.

Definition 4.9. [9] *Let* $f_R : (A,T) \to (B,T')$ *be an intuitionistic fuzzy mapping. Then* f_R *is called :*

1. *an intuitionistic fuzzy precontinuous mapping if* $f_R^{-1}(U')$ *is an IFPOS on A for every IFOS* U' *on B;*
2. *an intuitionistic fuzzy* α*-continuous mapping if* $f_R^{-1}(U')$ *is an IF*α*OS on A for every IFOS* U' *on B.*

The following proposition shows the relationship between intuitionistic fuzzy continuous mapping and intuitionistic fuzzy α-continuous mapping.

Proposition 4.10. *Let* $f_R : (A,T) \to (B,T')$ *be an intuitionistic fuzzy mapping. If* f_R *is an intuitionistic fuzzy continuous mapping, then* f_R *is an intuitionistic fuzzy* α*-continuous mapping.*

Proof. Let U' be an IFOS in B and we need to show that $f_R^{-1}(U')$ is an IFαOS in A. The fact that f_R is an intuitionistic fuzzy continuous mapping implies that $f_R^{-1}(U')$ is an IFOS in A. From Definition 3.11, it follows that

$$\mu_{f_R^{-1}(U')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x\in f^{-1}(supp(U')) \\ 0\ ,\ Otherwise\,, \end{cases}$$

and

$$\nu_{f_R^{-1}(U')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x\in f^{-1}(supp(U')) \\ 1\ ,\ Otherwise. \end{cases}$$

We conclude that, $f_R^{-1}(U')$ is an IFαOS in A. Hence, f_R is an intuitionistic fuzzy α-continuous mapping. □

Remark 4.11. *The converse of the above implication does not necessarily hold. Indeed, let us consider the intuitionistic fuzzy mapping* f_R *given in Example 4.7 and* T' *be an IF-topology given by* $T' = \{0_\sim, B, U_2'\}$*, where:*

$$\mu_{U_2'}(y) = \begin{cases} 0.3\ ,\ if\ y\geq -\frac{1}{2} \\ 0\ ,\ Otherwise\,, \end{cases} \qquad and\ \ \nu_{U_2'}(y) = \begin{cases} 0.4\ ,\ if\ y\geq -\frac{1}{2} \\ 1\ ,\ Otherwise. \end{cases}$$

It is easy to verify that

$$\mu_{f_R^{-1}(U_2')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_2')) \\ 0\ ,\ Otherwise\,, \end{cases} \qquad = \begin{cases} 0.55\ ,\ if\ x\in\mathbb{R}\setminus[-\frac{\sqrt{2}}{2}-1,\frac{\sqrt{2}}{2}-1] \\ 0\ ,\ Otherwise\,, \end{cases}$$

and

$$\nu_{f_R^{-1}(U_2')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x\in f^{-1}(supp(U_2')) \\ 1\ ,\ Otherwise\,, \end{cases} \qquad = \begin{cases} 0.4\ ,\ if\ x\in\mathbb{R}\setminus[-\frac{\sqrt{2}}{2}-1,\frac{\sqrt{2}}{2}-1] \\ 1\ ,\ Otherwise\,, \end{cases}$$

Hence, $int(f_R^{-1}(U_2')) = U_1$ *and* $cl(U_1) = 1_\sim$ *and* $int(1_\sim) = A$*. Thus,* $f_R^{-1}(U_2') \subseteq int(cl(int(f_R^{-1}(U_2'))))$*. We conclude that* $f_R^{-1}(U_2')$ *is an IF*α*S but not IFOS and* f_R *is an intuitionistic fuzzy* α*-continuous but not an intuitionistic fuzzy continuous.*

The following proposition shows the relationship between intuitionistic fuzzy α-continuous mapping and intuitionistic fuzzy pre-continuous mapping.

Proposition 4.12. *Let $f_R : (A,T) \to (B,T')$ be an intuitionistic fuzzy mapping. If f_R is an intuitionistic fuzzy α-continuous mapping, then f_R is an intuitionistic fuzzy pre-continuous mapping.*

Proof. Let U' be an IFOS in B and we need to show that $f_R^{-1}(U')$ is an IFPOS in A. The fact that f_R is an intuitionistic fuzzy α-continuous mapping implies that $f_R^{-1}(U')$ is an IFαOS in A. From Definition 3.11, it follows that

$$\mu_{f_R^{-1}(U')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x \in f^{-1}(supp(U')) \\ 0\ ,\ Otherwise\,, \end{cases}$$
$$\nu_{f_R^{-1}(U')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x \in f^{-1}(supp(U')) \\ 1\ ,\ Otherwise\,, \end{cases}$$

We conclude that, $f_R^{-1}(U')$ is an IFPOS in A. Hence, f_R is an intuitionistic fuzzy pre-continuous mapping. □

Remark 4.13. *The converse of the above implication is not necessarily holds. Indeed, let us consider the intuitionistic fuzzy mapping f_R given in Example 4.7 and T' be an IF-topology given by $T' = \{0_\sim, B, U_3'\}$, where:*

$$\mu_{U_3'}(y) = \begin{cases} 0.3\ ,\ if\ y \in [-1,0] \\ 0\ ,\ Otherwise\,, \end{cases} \quad and \quad \nu_{U_3'}(y) = \begin{cases} 0.4\ ,\ if\ y \in [-1,0] \\ 1\ ,\ Otherwise. \end{cases}$$

It is easy to verify that

$$\mu_{f_R^{-1}(U_3')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x \in f^{-1}(supp(U_3')) \\ 0\ ,\ Otherwise\,, \end{cases} = \begin{cases} 0.55\ ,\ if\ x \in [-2,0] \\ 0\ ,\ Otherwise\,, \end{cases}$$

and

$$\nu_{f_R^{-1}(U_3')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x \in [-2,0] \\ 1\ ,\ Otherwise\,, \end{cases} = \begin{cases} 0.4\ ,\ if\ x \in [-2,0] \\ 1\ ,\ Otherwise. \end{cases}$$

Hence, $cl(f_R^{-1}(U_3')) = 1_\sim$ and $int(1_\sim) = A$. Thus, $f_R^{-1}(U_3') \subseteq int(cl(f_R^{-1}(U_3')))$. We conclude that $f_R^{-1}(U_3')$ is an IFPOS and f_R is an intuitionistic fuzzy pre-continuous but not an intuitionistic fuzzy α-continuous.

5. Conclusion

In this work, the notion of intuitionistic fuzzy mapping based on the intuitionistic fuzzy relation as a generalization of the notion of fuzzy mapping defined by Ismail and Massa'deh is introduced and the most interesting properties are investigated. As applications, some properties of intuitionistic fuzzy continuous mappings in intuitionistic fuzzy topological space are provided and relations among various kinds of intuitionistic fuzzy continuity are investigated.

Future work is anticipated in multiple directions. We think it makes sense to study the notion of intuitionistic fuzzy mapping for other types of topologies based on the intuitionistic fuzzy sets. Moreover, we intend to extend this work to other kinds of intuitionistic fuzzy continuous mappings.

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