Intuitionistic Fuzzy Complete Lattices

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Abstract. In this paper, the concept of intuitionistic complete lattices is introduced. Some characterizations of such intuitionistic complete lattices are given. The Tarski-Davis fixed point theorem for intuitionistic fuzzy complete lattices is proved, which establish an other criterion for completeness of intuitionistic fuzzy complete lattices in terms of fixed points of intuitionistic monotone maps.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy order, Intuitionistic fuzzy complete lattice, Tarski-Davis fixed point theorem.

1 Introduction

The notion of a fuzzy set was first introduced by Zadeh [25] by assuming the standard negation that the non-membership degree is equal to one minus membership degree and this makes the fuzzy sets compliment. In logical area, membership degree and non-membership degree can be interpreted as positive and negative. This means that if the membership is correct, then the non membership is wrong. Obviously it explains that the contraries relation exists.
In 1983 Atanassov [1] proposed a generalization of Zadeh non-membership degree and introduced the notion of intuitionistic fuzzy set (A-IFSs for short). The non-membership degree used for Atanassov's intuitionistic fuzzy set is a more-or-less independent degree: the only condition is that the non-membership degree is less or equal to the one minus membership degree. Certainly fuzzy sets are intuitionistic fuzzy sets, but not conversely.
Inspired by the notion of intuitionistic fuzzy set, Burillo and Bustince ([12], [13]) introduced intuitionistic fuzzy relations as a natural generalizations of fuzzy relations. Intuitionistic fuzzy relations theory has been applied to many different fields, such as decision making, mathematical modelling, medical diagnosis, machine learning and market prediction, etc.
One of the important problems of fuzzy and intuitionistic fuzzy ordered set is to obtain an appropriate concepts of particular elements on a such structure like maximum, supremum, maximal elements and their duals, in particular, specific subclasses of fuzzy and intuitionistic fuzzy ordered sets. Several theoretical and applicational results connected with this problem can be found, e.g. in Bělohlávek [9], Bodenhofer and Klawonn [10], Bustince and Burillo ([15], [16]), Coppola et

al. [17], Tripathy et al. [23], Zadeh [26], Zhang et al. [29].
In this paper, according to the intuitionistic fuzzy order introduced by Burillo and Bustince ([12], [13]) and based on the notions of supremum and infimum of subsets on a universe X with respect to an intuitionistic fuzzy order defined on it introduced by Tripathy et al. [23], we propose a notion of an intuitionistic fuzzy complete lattice which is a generalization of the crisp complete lattice notion. Some characterizations of such intuitionistic fuzzy complete lattice expressed in terms of supremum, infimum, chains and maximal chains are given.
One of the consequence of Tarski and Davis fixed point theorems in crisp lattices ([18], [22]) is that they established a criterion for completeness of lattices in terms of fixed points of monotone maps. In the last section we will focus on this criterion and Tarski-Davis fixed point theorem for intuitionistic fuzzy complete lattices will be proved.

2 Preliminaries

This section contains the basic definitions and properties of intuitionistic fuzzy sets, intuitionistic fuzzy relations, intuitionistic fuzzy lattices and some related notions that will be needed in the next sections of this paper. At first we recall some basic concepts of intuitionistic fuzzy sets. More details can be found in ([1] - [8], [11], [20], [27]).
Let X be a universe, then a fuzzy set $A = \{\langle x, \mu_A(x)\rangle \ / \ x \in X\}$ defined by Zadeh [25] is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of a membership of the element x in the fuzzy subset A for each $x \in X$.
In [1] Atanassov introduced another fuzzy object, called intuitionistic fuzzy set (briefly IFS or A-IFS) as a generalization of the concept of fuzzy set, shown as follows

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \ / \ x \in X\},$$

which is characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1\,, \tag{1}$$

for any $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy set A for each $x \in X$.
In the fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard negation) and thus it is fixed. In intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$, but not conversely.

Definition 1. *Let A be an intuitionistic fuzzy set on universe X, the support of A is the crisp subset of X given by*
$Supp(A) = \{x \in X \ / \ \mu_A(x) > 0 \ \textit{or} \ (\mu_A(x) = 0 \ \textit{and} \ \nu_A(x) < 1)\}$.

An intuitionistic fuzzy relation from a universe X to a universe Y is an intuitionistic fuzzy subset in $X \times Y$, i.e. is an expression R given by
$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \ / \ (x,y) \in X \times Y\}$, where $\mu_R : X \times Y \to [0,1]$, $\nu_A : X \times Y \to [0,1]$ with the condition

$$0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1\,, \tag{2}$$

for any $(x,y) \in X \times Y$. The value $\mu_R(x,y)$ is called the degree of a membership of (x,y) in R and $\nu_R(x,y)$ is called the degree of a non-membership of (x,y) in R.

Next, we need the following definitions.
Let R be an intuitionistic fuzzy relation from a universe X to a universe Y. The transposition R^t of R is the intuitionistic fuzzy relation from the universe Y to the universe X defined by

$$R^t = \{\langle (x,y), \mu_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\},$$

where $\mu_{R^t}(x,y) = \mu_R(y,x)$ and $\nu_{R^t}(x,y) = \nu_R(y,x)$, for any $(x,y) \in X \times Y$.

Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y. R is said to be contained in P or we say that P contains R (notation $R \subseteq P$) if for all $(x,y) \in X \times Y$ $\mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.

The intersection (resp. the union) of two intuitionistic fuzzy relations R and P from a universe X to a universe Y is defined as

$$R \bigcap P = \{\langle (x,y), \mu_{R\cap P}(x,y), \nu_{R\cap P}(x,y)\rangle\},$$

where $\mu_{R\cap P}(x,y) = \min(\mu_R(x,y), \mu_P(x,y))$ and $\nu_{R\cap P}(x,y) = \max(\nu_R(x,y), \nu_P(x,y))$ for any $(x,y) \in X \times Y$.

The union of two intuitionistic fuzzy relations R and P from a universe X to a universe Y defined as

$$R \bigcup P = \{\langle (x,y), \mu_{R\cup P}(x,y), \nu_{R\cup P}(x,y)\rangle\},$$

where $\mu_{R\cup P}(x,y) = \max(\mu_R(x,y), \mu_P(x,y))$ and $\nu_{R\cup P}(x,y) = \min(\nu_R(x,y), \nu_P(x,y))$ for any $(x,y) \in X \times Y$.

In general, if A is a set of intuitionistic fuzzy relations from a universe X to a universe Y, then

$$\bigcap_{R\in A} R = \{\langle (x,y), \mu_{\cap_{R\in A} R}(x,y), \nu_{\cap_{R\in A} R}(x,y)\rangle\},$$

where $\mu_{\cap_{R\in A} R}(x,y) = \inf_{R\in A} \mu_R(x,y)$ and $\nu_{\cap_{R\in A} R}(x,y) = \sup_{R\in A} \nu_R(x,y)$ for any $(x,y) \in X \times Y$;

$$\bigcup_{R\in A} R = \{\langle (x,y), \mu_{\cup_{R\in A} R}(x,y), \nu_{\cup_{R\in A} R}(x,y)\rangle\},$$

where $\mu_{\cup_{R\in A}R}(x,y) = \sup_{R\in A}\mu_R(x,y)$ and $\nu_{\cup_{R\in A}R}(x,y) = \inf_{R\in A}\nu_R(x,y)$ for any $(x,y) \in X \times Y$.

Let R be an intuitionistic fuzzy relation from a universe X to a universe X (intuitionistic fuzzy relation on a universe X, for short). The following properties are crucial in this paper (see e.g. [12] - [16], [19], [23], [24], [28]):

(i) Reflexivity: $\mu_R(x,x) = 1$ for any $x \in X$. Just notice that $\nu_R(x,x) = 0$ for any $x \in X$.

(ii) Antisymmetry: if for any $x, y \in X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases},$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) Perfect antisymmetry: if for any $x, y \in X$ with $x \neq y$
$\mu_R(x,y) > 0$ or ($\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1$) then $\mu_R(y,x) = 0$ and $\nu_R(y,x) = 1$.

(iv) Transitivity: $R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R$.

Remark 1. The definition of perfect antisymmetry given in (iii) is equivalent to the following one
for any $x, y \in X$, ($\mu_R(x,y) > 0$ and $\mu_R(y,x) > 0$) or ($\nu_R(x,y) < 1$ and $\nu_R(y,x) < 1$) implies that $x = y$.

The composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ in the above definition of transitivity means that

$$R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)],$$
$$\lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\rangle \mid x, z \in X\},$$

where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)] + \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)] \leq 1\,,$$

for any $x, z \in X$.

The properties of this composition and the choice of α, β, λ and ρ, for which this composition fulfills a maximal number of properties, are investigated in ([12], [13], [14], [15], [16], [19]). If no other conditions are imposed, in the sequel we will take $\alpha = \sup$, $\beta = \min$, $\lambda = \inf$ and $\rho = \max$.

Notice that in [15], Bustince and Burillo mentioned that the definition of intuitionistic antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic perfect antisymmetry does recover the definition of fuzzy antisymmetry given by Zadeh [26] when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order used in this paper.

Definition 2 ([12], [13]). *Let X be a nonempty crisp set and* $R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid x, y \in X\}$ *be an intuitionistic fuzzy relation on* X. *R is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.*

A nonempty set X with an intuitionistic fuzzy order R defined on it is called an intuitionistic fuzzy ordered set and we denote it by (X, μ_R, ν_R).

Notice that any partially ordered set $(X, \leq)$ and generally any fuzzy ordered set (X, R) can be regarded as intuitionistic fuzzy ordered sets.

Example 1. Let $m, n \in N$. Then, the intuitionistic fuzzy relation R defined for all $m, n \in N$ by

$$\mu_R(m,n) = \begin{cases} 1\ ,\ if\quad m = n \\ 1 - \frac{m}{n}\ ,\quad if\quad m < n \\ 0\ ,\quad if\quad m > n \end{cases},$$

and

$$\nu_R(m,n) = \begin{cases} 0\ ,\ if\quad m = n \\ \frac{m}{2n}\ ,\quad if\quad m < n \\ 1\ ,\quad if\quad m > n \end{cases}$$

is an intuitionistic fuzzy order on N.

On the basis of the above definition of perfect antisymmetry we define linear or total intuitionistic fuzzy order as follows

Definition 3. *An intuitionistic fuzzy order R on a universe X is linear (or total) if for every $x, y \in X$* $[\mu_R(x,y) > 0$ *or* $(\mu_R(x,y) = 0$ *and* $\nu_R(x,y) < 1)]$ *or*
$[\mu_R(y,x) > 0$ *or* $(\mu_R(y,x) = 0$ *and* $\nu_R(y,x) < 1)]$.

Definition 4. *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain.*

For an intuitionistic fuzzy ordered set (X, μ_R, ν_R) and $x \in X$, the intuitionistic fuzzy sets $R_{\geq[x]}$ and $R_{\leq[x]}$ defined in X by
$R_{\geq[x]} = \{\langle y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)\rangle\ /\ y \in X\}$, where $\mu_{R_{\geq[x]}}(y) = \mu_R(x,y)$ and $\nu_{R_{\geq[x]}}(y) = \nu_R(x,y)$.

$R_{\leq[x]} = \{\langle y, \mu_{R_{\leq[x]}}(y), \nu_{R_{\leq[x]}}(y)\rangle\ /\ y \in X\}$, where $\mu_{R_{\leq[x]}}(y) = \mu_R(y,x)$ and $\nu_{R_{\leq[x]}}(y) = \nu_R(y,x)$. $R_{\geq[x]}$ and $R_{\leq[x]}$ are called the dominating class of x and the class dominated by x, respectively.

Remark 2. The notions of the dominating class of x and the class dominated by x are generalizations of the classical notions $\uparrow x$ and $\downarrow x$ in a usual poset.

Next, we recall the definition of upper bounds, lower bounds, supremum and infimum on intuitionistic fuzzy ordered sets.

Definition 5 ([23]). *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X.*

(i) The set of upper bounds of A with respect to R is the intuitionistic fuzzy subset of X defined by

$$U(R,A)(y) = \bigcap_{x\in A} R_{\geq[x]}(y) \tag{3}$$

for any $y \in X$;

(ii) The set of lower bounds of A with respect to R is the intuitionistic fuzzy subset of X defined by

$$L(R,A)(y) = \bigcap_{x\in A} R_{\leq[x]}(y) \tag{4}$$

for any $y \in X$.

Definition 6 ([23]). *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X. An element $x \in X$ is called the least upper bound (or a supremum) of A with respect to R if*

(i) $x \in Supp(U(R,A))$ and

(ii) for all other $y \in Supp(U(R,A))$, $\mu_R(x,y) > 0$ or ($\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1$).

An element $x \in X$ is called the greatest lower bound (or an infimum) of A with respect to R if

(i) $x \in Supp(L(R,A))$ and

(ii) for all other $y \in Supp(L(R,A))$, $\mu_R(y,x) > 0$ or ($\mu_R(y,x) = 0$ and $\nu_R(y,x) < 1$).

Remark 3. Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X. If the supremum and the infimum of A with respect to R exist, then from the perfect antisymmetry of R they are unique and denoted by $\sup_R(A)$ and $\inf_R(A)$, respectively.

Definition 7. *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) is called an intuitionistic fuzzy ordered lattice with respect to the intuitionistic fuzzy order R (or simply, intuitionistic fuzzy lattices) if each pair of elements $\{x, y\}$ of X has a supremum and an infimum.*

Next, we introduce the notion of intuitionistic fuzzy complete lattices which is a natural generalization of the notion of crisp complete lattices.

Definition 8. *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) is called an intuitionistic fuzzy complete lattice if $\sup_R(A)$ and $\inf_R(A)$ exist for every nonempty subset $A \subseteq X$.*

Definition 9. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set.*

(i) An element $\top \in X$ is called the greatest element (the maximum) of X with respect to R or the intuitionistic fuzzy maximum of X if $\mu_R(x, \top) > 0$ *or* $(\mu_R(x, \top) = 0$ *and* $\nu_R(x, \top) < 1)$ *for all* $x \in X$.

(ii) An element $\perp \in X$ is called the least element (the minimum) of X with respect to R or the intuitionistic fuzzy minimum if $\mu_R(\perp, x) > 0$ *or* $(\mu_R(\perp, x) = 0$ *and* $\nu_R(\perp, x) < 1)$ *for all* $x \in X$.

Remark 4. Every intuitionistic fuzzy complete lattice must have a greatest element (or a maximum) and a least element (or a minimum). The greatest element will be denoted $\top_X$ and the least element $\perp_X$. It is easily follows that

$$\top_X = \sup_R(X) = \inf_R(\emptyset) \text{ and } \perp_X = \inf_R(X) = \sup_R(\emptyset) \ .$$

In the last section we will need the following definitions.

Definition 10. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set. Then a map $f : X \to X$ is called intuitionistic fuzzy monotone if $\mu_R(f(x), f(y)) \geq \mu_R(x, y)$ and $\nu_R(f(x), f(y)) \leq \nu_R(x, y)$ for all $x, y \in X$.*

Definition 11. *An element $x \in X$ is called a fixed point of a map $f : X \to X$ if $f(x) = x$. The set of all fixed points of f will be denoted by $Fix(f)$.*

3 Characterizations of intuitionistic fuzzy complete lattices

In this section we will provide an interesting characterization of intuitionistic fuzzy complete lattices in terms of supremum and infimum of its subsets, as well as in terms of its intuitionistic fuzzy chains and maximal intuitionistic fuzzy chains.

The following lemma is immediate.

Lemma 1. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set, A be a subset of X and $x \in X$. Then, it holds that*

(i) $x = \sup_R(A)$ *with respect to R if and only if* $x = \inf_{R^t}(A)$ *with respect to* R^t;

(ii) $x = \inf_R(A)$ *with respect to R if and only if* $x = \sup_{R^t}(A)$ *with respect to* R^t;

(iii) (X, μ_R, ν_R) *is intuitionistic fuzzy complete lattice if and only if* $(X, \mu_{R^t}, \nu_{R^t})$ *is intuitionistic fuzzy complete lattice.*

Theorem 1. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set. Then, it holds that*

(i) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if* $\sup_R(A)$ *exists for all* $A \subseteq X$;

(ii) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if* $\inf_R(A)$ *exists for all* $A \subseteq X$.

Proof. Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and $A \subseteq X$.

(i) It is obvious that if (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, then $\sup_R(A)$ exists for all $A \subseteq X$.
Conversely, suppose that $\sup_R(A)$ exists for all $A \subseteq X$ and we will show that every nonempty subset $A \subseteq X$ has an infimum. Let $L(R, A)$ be the intuitionistic fuzzy set of lower bounds of A with respect to R. Then, it holds that $\sup_R(Supp(L(R, A)))$ exists. Setting that $m = \sup_R(Supp(L(R, A)))$ and we show that $m = \inf_R(A)$. First, since $m \in Supp(U(R, Supp(L(R, A))))$, then it holds that $\mu_{U(R,Supp(L(R,A)))}(x) > 0$ or $[\mu_{U(R,Supp(L(R,A)))}(x) = 0$ and $\nu_{U(R,Supp(L(R,A)))}(x) < 1]$. By (3) we know that

$$U(R, Supp(L(R, A)))(y) = \bigcap_{x \in Supp(L(R,A))} R_{\geq [x]}(y).$$

Since $R_{\geq [x]} = \{\langle y, \mu_{R_{\geq [x]}}(y), \nu_{R_{\geq [x]}}(y)\rangle \ / \ y \in X\}$, where $\mu_{R_{\geq [x]}}(y) = \mu_R(x, y)$ and $\nu_{R_{\geq [x]}}(y) = \nu_R(x, y)$ and by the fact that $R_{\geq [x]} = R^t_{\leq [x]}$ and $U(R, A) = L(R^t, A)$, it follows that

$$U(R, Supp(L(R, A)) = L(R^t, Supp(L(R, A))) = L(R, A).$$

Hence,
$$m \in Supp(L(R, A)).$$

In the same way, for all $y \in Supp(L(R, A))$, it holds that $\mu_R(y, m) > 0$ or $(\mu_R(y, m) = 0$ and $\nu_R(y, m) < 1)$.
Thus $m = \inf_R(A)$, which implies that $\inf_R(A)$ exists. Therefore, (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

(ii) Follows from Lemma 1 and (i).

Remark 5. In the above Theorem 1 the existence of $\inf_R(\emptyset)$ guarantees the greatest element of (X, μ_R, ν_R), and in similar way, the existence of $\sup_R(\emptyset)$ guarantees the least element of (X, μ_R, ν_R). So an equivalent formulation of Theorem 1 can be written in the following way

(i) (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice if and only if it has the least element and $\sup_R(A)$ exists for all nonempty $A \subseteq X$;
(ii) (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice if and only if it has the greatest element and $\inf_R(A)$ exists for all nonempty $A \subseteq X$.

Theorem 2. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. Then the following are equivalent:*

(i) (X, μ_R, ν_R) *is intuitionistic fuzzy complete lattice;*
(ii) (X, μ_R, ν_R) *is intuitionistic fuzzy chain-complete (i.e. every nonempty intuitionistic fuzzy chain in* (X, μ_R, ν_R) *has a supremum and an infimum);*
(iii) Every maximal intuitionistic fuzzy chain of X is an intuitionistic fuzzy complete lattice.

Proof. (i)⇒ (ii) is obvious.
To prove (ii)⇒ (iii), let C be a maximal intuitionistic fuzzy chain (with respect to set inclusion) of X.
First, we will show that C has an intuitionistic fuzzy maximum and an intuitionistic fuzzy minimum. Since C is an intuitionistic fuzzy chain in (X, μ_R, ν_R) then it holds from (ii) that C has a supremum and an infimum. By using the fact that C is maximal (with respect to set inclusion) we obtain that $c_1 = \sup_R(C)$ is the maximum and $c_2 = \inf_R(C)$ is the minimum.
Second, let $A \subseteq C$. Since $A \subseteq C$ then it holds that A is an intuitionistic fuzzy chain. By (ii) we know that $\sup_R(A)$ exists in (X, μ_R, ν_R) and we denoted it by m. Now, it suffices to show that $m \in C$. Suppose that $m \notin C$, then it follows three cases:

(a) If $[\mu_R(x,m) > 0$ or $(\mu_R(x,m) = 0$ and $\nu_R(x,m) < 1]$ or $[\mu_R(m,x) > 0$ or $(\mu_R(m,x) = 0$ and $\nu_R(m,x) < 1]$ for all $x \in C$, then $C \cup \{m\}$ is intuitionistic fuzzy chain in (X, μ_R, ν_R). This is a contradiction with the fact that C is maximal.

(b) If there exist $x \in C$ such that $[\mu_R(x,m) = 0$ and $\nu_R(x,m) = 1]$, then it holds from the transitivity of R that

$$\mu_R(x, c_1) \wedge \mu_R(c_1, m) \leq \mu_R(x, m)$$

and

$$\nu_R(x, c_1) \vee \nu_R(c_1, m) \geq \nu_R(x, m).$$

Since $[\mu_R(x,m) = 0$ and $\nu_R(x,m) = 1]$, then it holds that $\mu_R(c_1,m) = 0$ and $\nu_R(c_1,m) = 1$. Hence $\sup_R\{c_1, m\} \notin C$. Thus, $C \bigcup \{\sup_R\{c_1, m\}\}$ is an intuitionistic fuzzy chain, which is a contradiction with maximality of C.

(c) If there exist $x \in C$ such that $[\mu_R(m,x) = 0$ and $\nu_R(m,x) = 1]$, then it follows similarly as (b).

As consequence of the above cases we get $m \in C$. Thus, A has a supremum in C. Therefore, C is an intuitionistic fuzzy complete lattice which follows from Theorem 1.
(iii)⇒ (i) Suppose that every maximal intuitionistic fuzzy chain of X is an intuitionistic fuzzy complete lattice and we will show that (X, μ_R, ν_R) is intuitionistic fuzzy complete lattice.
Let $A \subseteq X$ and $\mathcal{BFC}(Supp(U(R, A)))$ denote the set of all intuitionistic fuzzy fuzzy chains $C \subseteq Supp(U(R, A))$, ordered in classical way by $C_1 \sqsubseteq C_2$ if and only f C_1 is an intuitionistic fuzzy filter of C_2. This means that $C_1 \subseteq C_2$ or [if $x \in C_1$ and $y \in C_2$ with $\mu_R(x,y) > 0$ or $(\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1)$ then $y \in C_1$].
Next, let $\{C_i : i \in I \subseteq N\}$ be a chain of $\mathcal{BFC}(Supp(U(R, A)))$ under the crisp order defined above. On the one hand, since C_i is an intuitionistic fuzzy chain of $Supp(U(R, A))$ and $C_i \subseteq C_{i+1}$ for all $i \in I$, then $\bigcup_{i \in I} C_i$ is an intuitionistic fuzzy chain of $Supp(U(R, A))$. Hence, $\bigcup_{i \in I} C_i \in \mathcal{BFC}(Supp(U(R, A)))$. On the

other hand, $\bigcup_{i\in I} C_i$ is an upper bound of $\{C_i\}_{i\in I}$.
By Zorn's Lemma, we know that $\mathcal{BFC}(Supp(U(R,A)))$ has a maximal element denoted by C_m with respect to the above crisp order $\sqsubseteq$.
Let K be a maximal intuitionistic fuzzy chain such that $C_m \subseteq K$. By hypothesis, K is an intuitionistic fuzzy complete lattice, which implies that C_m has a an infimum denoted by c in (K, μ_R, ν_R).
Now, we will show that $c = \sup_R(A)$. Indeed, let $x \in A$. Since $C_m \subseteq Supp(U(R,A))$, then it holds that $\mu_R(x,y) > 0$ or ($\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1$) for all $y \in C_m$. Hence $\mu_R(x,c) > 0$ or ($\mu_R(x,c) = 0$ and $\nu_R(x,c) < 1$). Thus, $c \in Supp(U(R,A))$. For all other $y \in Supp(U(R,A))$, it holds that $\mu_R(c,y) > 0$ or ($\mu_R(c,y) = 0$ and $\nu_R(c,y) < 1$). Otherwise, we get a contradiction with the maximality of C_m. Thus, $c = \sup_R(A)$. Now, (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice follows from Theorem 1(i).

4 Tarski-Davis fixed point theorem for intuitionistic fuzzy complete lattices

Tarski and Davis in their results established a criterion for completeness of lattices in terms of fixed points of monotone maps. In the last section, we will show that this criterion also stay valid for intuitionistic fuzzy complete lattices.
Additionally we need the following definition

Definition 12. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and $\{a_i\}_{i\in I\subseteq N}$ be a subset of elements of X. Then*

(i) $\{a_i\}_{i\in I\subseteq N}$ is called an intuitionistic fuzzy ascending chain (or an ascending chain with respect to R) if $\mu_R(a_i, a_{i+1}) > 0$ or ($\mu_R(a_i, a_{i+1}) = 0$ and $\nu_R(a_i, a_{i+1}) < 1$), for all $i \in I$. Descending intuitionistic fuzzy chain (or an descending chain with respect to R) defined dually.

(ii) (X, μ_R, ν_R) is said to be satisfy the intuitionistic fuzzy ascending chain condition (or the ACC_R, for short) if every intuitionistic fuzzy ascending chain $\{a_i\}_{i\in I\subseteq N}$ of elements of X is eventually stationary (i.e. there exist a positive integer $n \in I$ such that $a_m = a_n$ for all $m > n$). In other words, (X, R) contains no infinite intuitionistic fuzzy ascending chain.

(iii) Similarly, (X, μ_R, ν_R) is said to be satisfy the intuitionistic fuzzy descending chain condition (or the DCC_R, for short) if every intuitionistic fuzzy descending chain $\{a_i\}_{i\in I\subseteq N}$ of elements of X is ultimately stationary.

From Theorem 1 and Theorem 2, we derive the following results (unfortunately without very extensive proofs).

Proposition 1. *Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. If (X, μ_R, ν_R) is not intuitionistic fuzzy complete lattice, then there exists an intuitionistic fuzzy chain C satisfies the ACC_R and has no infimum and an intuitionistic fuzzy chain D satisfies the DCC_R and has no supremum, such that* *(i)* $\mu_R(d,c) > 0$ *or* $(\mu_R(d,c) = 0$ *and* $\nu_R(d,c) < 1)$ *for any* $d \in D$ *and* $c \in C$ *;*

(ii) *For all* $x \in X$*, either there exists* $c \in C$ *with* $(\mu_R(x,c) = 0$ *and* $\nu_R(x,c) = 1)$ *or there exists* $d \in D$ *with* $(\mu_R(d,x) = 0$ *and* $\nu_R(d,x) = 1)$*, i.e. there is no element* $x \in X$ *such that* $x \in Supp(L(R,C)) \bigcap Supp(U(R,D))$*.*

Theorem 3. *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy lattice. Then* (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if every intuitionistic fuzzy monotone map* $f : X \to X$ *has a fixed point. In this situation, the subset* $Fix(f)$ *of fixed point of* f *is an intuitionistic fuzzy complete lattice.*

5 Conclusion

In this paper we have introduced the notion of intuitionistic fuzzy complete lattice and investigated its most interesting properties. Some characterizations of intuitionistic fuzzy complete lattice expressed in terms of supremum, infimum, chains and maximal chains are given. Moreover, the Tarski-Davis fixed point theorem for intuitionistic fuzzy complete lattices is presented (without the extensive proof), which establish an other criterion for completeness of intuitionistic fuzzy complete lattices in terms of fixed points of intuitionistic monotone maps.

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