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ORIGINAL ARTICLE

The Fixed Point Property for Intuitionistic Fuzzy Lattices

Lemnaouar Zedam · Soheyb Milles · Ewa Rak

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Abstract In this paper, based on the concept of intuitionistic fuzzy lattice previously introduced by Tripathy and his colleagues, a class of intuitionistic fuzzy complete lattices is proposed with some interesting characterizations given. In particular, we show the fixed point property for this proposed class. Conversely, we show that any intuitionistic fuzzy lattice is complete having its fixed point property. These results establish a criterion for completeness of intuitionistic fuzzy lattices in terms of the fixed points of their intuitionistic fuzzy monotone mappings.

Keywords Atanassov's intuitionistic fuzzy set · Intuitionistic fuzzy relation · Intuitionistic fuzzy lattice · Fixed point property

1. Introduction

The fixed point property is one of the most famous topics in order theory. Let P be an ordered set, P is said to have the fixed point property if every order-preserving

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map f of P into itself has a fixed point. One of the important roles of the fixed point property is to combine properties of a set with the properties of maps in that set. For the new structures the fixed point property can provide a familiar setting, helping us to investigate these structures.

Atanassov [1] introduced a new structure called intuitionistic fuzzy set or Atanassov's intuitionistic fuzzy set as it is also called by several authors. Atanassov's intuitionistic fuzzy set is a generalization of Zadeh's fuzzy set previously introduced in [27]. In fuzzy set theory, the non-membership degree of an element x can be viewed as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard strong negation) and this is fixed. While in the intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly, fuzzy sets are Atanassov's intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Based on Atanassov's intuitionistic fuzzy set, Burillo and Bustince [9, 10] introduced the concept of intuitionistic fuzzy relation, particulary, they introduced the intuitionistic fuzzy order or (intuitionistic fuzzy ordered set) as a natural generalization of fuzzy order relation previously introduced by Zadeh [28]. Intuitionistic fuzzy relations theory has been applied to many different fields, such as decision making, mathematical modeling, medical diagnosis, control systems, machine learning, market prediction, and so on.

One of the important problems of fuzzy and intuitionistic fuzzy ordered set is to obtain an appropriate concept of the particular elements on such structure, like upper bound, maximum, supremum, maximal elements and their duals, in order to obtain new structures and particular classes of fuzzy and intuitionistic fuzzy ordered sets. Several theoretical and applicational results connected with this problem can be found, e.g. in Bělohlávek [6], Bodenhofer and Klawonn [7], Bustince and Burillo [12, 13], Coppola et al. [14], Tripathy et al. [24] and Zadeh [28]. In particular, Tripathy and his colleagues [24] introduced the concepts of the upper bound, the supremum and their duals of subsets on a universe X with respect to an intuitionistic fuzzy order defined on it. Also, they introduced and studied a concept of lattice with respect to an intuitionistic fuzzy order defined on it. This concept is extensively used and discussed in the fuzzy and intuitionistic fuzzy settings by several authors [6, 19, 20, 23, 25, 31].

In a recent conference paper [21], based on the concept of intuitionistic fuzzy lattice previously proposed by Tripathy et al. [24], we introduced the notion of intuitionistic fuzzy complete lattice and investigated its basic characterizations. The introduced notion of intuitionistic fuzzy complete lattice is a generalization of the notion of crisp complete lattice. In this paper, we extend these characterizations by considering others completeness criterions. The characterizations of intuitionistic fuzzy complete lattices expressed in terms of the existence of the supremum or the infimum of their subsets, in terms of intuitionistic fuzzy chains and maximal chains and in terms of intuitionistic fuzzy ascending (resp. descending) chains are given. Furthermore, we will show that an intuitionistic fuzzy lattice X is complete if and only if any intuitionistic fuzzy monotone mapping $f : X \to X$ has a fixed point, i.e., an intuitionistic fuzzification of Tarski-Davis's fixed point theorem [15, 22]. This leads to see clearly that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa. These results show the key role of the fixed point property for establishing a completeness criterion for intuitionistic fuzzy lattices.

Note that many function spaces, in general, fuzzy or intuitionistic fuzzy function spaces can be viewed as intuitionistic fuzzy lattices. This fact allows the obtained results to be used for expressing mathematical problems in fuzzy and intuitionistic fuzzy function spaces. Particularly, the fixed point property for these spaces as intuitionistic fuzzy lattices can be employed to examine the theoretical solvability of linear, integral or differential equations and to develop numerical approaches for their solutions.

The contents of the paper are organized as follows. In Section 2, we recall basic concepts and properties of intuitionistic fuzzy sets and intuitionistic fuzzy relations that will be needed throughout this paper. In Section 3, we provide interesting characterizations of intuitionistic fuzzy complete lattices in terms of the existence of the supremums or the infimums of its subsets, as well as in terms of its intuitionistic fuzzy chains and maximal chains. A completeness criterion of intuitionistic fuzzy lattices in terms of the fixed point property is given in Section 4. We finish with some concluding remarks and future research in Section 5.

2. Preliminaries

This section contains the basic definitions and properties of intuitionistic fuzzy sets, intuitionistic fuzzy relations and some related notions that will be needed throughout this paper. The terminology, notions and notations used here are introduced in the following references: Atanassov [1, 2, 3, 4], Biswas [5], Burillo and Bustince [8, 9, 10, 11, 12, 13], Deschrijver and Kerre [16], Gerstenkorn and Manko [17], Grzegorzewski and Mrówka [18], Tripathy et al. [24], Xu [26] and Zedam et al. [29, 30].

2.1. Atanassov's Intuitionistic Fuzzy Sets

Let X be a universe. Then a fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ defined by Zadeh [27] is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of a membership of the element x in the fuzzy subset A for each $x \in X$. Atanassov in [1] introduced another fuzzy object, called intuitionistic fuzzy set as a generalization of the concept of fuzzy set, shown as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

which is characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$, with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{1}$$

for any $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy set A for each $x \in X$.

In the fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard negation) and thus it is fixed. In the intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$, but not conversely.

Definition 1 [4] *Let A be an intuitionistic fuzzy set on universe X. The support of A is the crisp subset of X given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

2.2. Intuitionistic Fuzzy Relations

Burillo and Bustince [9, 10] introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 2 [9, 10] *An intuitionistic fuzzy binary relation (an intuitionistic fuzzy relation, for short) from a universe X to a universe Y is an intuitionistic fuzzy subset of $X \times Y$, i.e., is an expression R given by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \to [0, 1], \text{ and } \nu_A : X \times Y \to [0, 1],$$

satisfy the condition

$$0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1 \tag{2}$$

for any $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of non-membership of (x, y) in R.

Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y. R is said to be contained in P or we say that P contains R, denoted by $R \subseteq P$, if for all $(x, y) \in X \times Y$ it holds that $\mu_R(x, y) \leq \mu_P(x, y)$ and $\nu_R(x, y) \geq \nu_P(x, y)$. The transpose (or the inverse) R^t of R is the intuitionistic fuzzy relation from the universe Y to the universe X defined by $R^t = \{\langle (x, y), \mu_{R^t}(x, y), \nu_{R^t}(x, y)\rangle \mid (x, y) \in X \times Y\}$, where $\mu_{R^t}(x, y) = \mu_R(y, x)$ and $\nu_{R^t}(x, y) = \nu_R(y, x)$ for any $(x, y) \in X \times Y$. The intersection of two intuitionistic fuzzy relations R and P from a universe X to a universe Y is defined as:

$$R \cap P = \{\langle (x, y), \min(\mu_R(x, y), \mu_P(x, y)), \max(\nu_R(x, y), \nu_P(x, y))\rangle \mid (x, y) \in X \times Y\}.$$

The union of two intuitionistic fuzzy relations R and P from a universe X to a universe Y is defined as:

$$R \cup P = \{\langle (x, y), \max(\mu_R(x, y), \mu_P(x, y)), \min(\nu_R(x, y), \nu_P(x, y))\rangle \mid (x, y) \in X \times Y\}.$$

Let R be an intuitionistic fuzzy relation from a universe X into itself (intuitionistic fuzzy relation on X, for short). The following properties are crucial in this paper (see e.g., [9, 10, 24, 30]):

(i) *Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.

(ii) *Antisymmetry*: for any $x, y \in X$, $x \neq y$, then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x)\,, \\ \nu_R(x,y) \neq \nu_R(y,x)\,, \\ \pi_R(x,y) = \pi_R(y,x)\,. \end{cases}$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) *Perfect antisymmetry*: for any $x, y \in X$ with $x \neq y$ and

$$\begin{cases} \mu_R(x,y) > 0\,, \\ \quad \text{or} \\ \mu_R(x,y) = 0 \ \text{ and } \nu_R(x,y) < 1\,, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0\,, \\ \quad \text{and} \\ \nu_R(y,x) = 1\,. \end{cases}$$

(iv) *Transitivity*: $R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R$.

In the above definition, the composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ used in the transitivity means that $R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y),\mu_R(y,z)]\},\ \lambda_{y\in X}\{\rho[\nu_R(x,y),\nu_R(y,z)]\}\rangle \mid x,z \in X\}$, where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y),\mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y),\nu_R(y,z)]\} \leq 1$$

for any $x, z \in X$. The properties of this composition and the choice of α, β, λ and ρ, for which this composition fulfills a maximal number of properties, are investigated in [9]-[13], [16]. If no other conditions are imposed, in the following we will take $\alpha = \sup$, $\beta = \min$, $\lambda = \inf$ and $\rho = \max$.

Note that Bustince and Burillo in [12], mentioned that the definition of intuitionistic fuzzy antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic fuzzy perfect antisymmetry recovers the definition of fuzzy antisymmetry given by Zadeh [28] when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order used in this paper.

Definition 3 [9, 10] *Let X be a nonempty crisp set and R be an intuitionistic fuzzy relation on X. R is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.*

A nonempty set X with an intuitionistic fuzzy order R defined on it is called an intuitionistic fuzzy ordered set and is denoted by (X, μ_R, ν_R). It easily follows that any partially ordered set $(X, \leq)$ and any fuzzy ordered set (X, R) can be viewed as intuitionistic fuzzy ordered sets.

Example 1 Let $m, n \in \mathbb{N}$. Then, the following intuitionistic fuzzy relation R on $\mathbb{N}$ is an intuitionistic fuzzy order:

$$\mu_R(m,n) = \begin{cases} 1\,,\ if\ m = n \\ 1 - \frac{m}{n}\,,\ \ if\ \ m < n \\ 0\,,\ if\ \ m > n \end{cases} \text{ and } \nu_R(m,n) = \begin{cases} 0\,,\ if\ \ m = n\,, \\ \frac{m}{2n}\,,\ if\ \ m < n\,, \\ 1\,,\ if\ \ m > n\,. \end{cases}$$

On the basis of the above definition of perfect antisymmetry we define a complete (or total) intuitionistic fuzzy order as follows.

Definition 4 [30] *An intuitionistic fuzzy order R on a universe X is called complete (or total), if, for any $x, y \in X$, it holds that*

$$[\mu_R(x, y) > 0 \text{ or } (\mu_R(x, y) = 0 \text{ and } \nu_R(x, y) < 1)],$$

or

$$[\mu_R(y, x) > 0 \text{ or } (\mu_R(y, x) = 0 \text{ and } \nu_R(y, x) < 1)].$$

Definition 5 [30] *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain.*

3. Intuitionistic Fuzzy Complete Lattices

In this section, we recall the notion of intuitionistic fuzzy complete lattice firstly introduced in [21] and some basic characterizations in terms of the existence of the supremum and the infimum of its subsets, and in terms of its intuitionistic fuzzy chains and maximal intuitionistic fuzzy chains are shown. These basic characterizations will be used as auxiliary results in the next section to show that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa.

3.1. Intuitionistic Fuzzy Lattices

In this subsection, we recall some basic concepts related to intuitionistic fuzzy lattice. Further information can be found in [24].

For an intuitionistic fuzzy ordered set (X, μ_R, ν_R) and $x \in X$, the intuitionistic fuzzy sets $R_{\geq[x]}$ and $R_{\leq[x]}$ defined in X by

$$R_{\geq[x]} = \{\langle y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)\rangle \mid y \in X\}\,,$$

where $\mu_{R_{\geq[x]}}(y) = \mu_R(x, y)$ and $\nu_{R_{\geq[x]}}(y) = \nu_R(x, y)$.

$$R_{\leq[x]} = \{\langle y, \mu_{R_{\leq[x]}}(y), \nu_{R_{\leq[x]}}(y)\rangle \mid y \in X\},$$

where $\mu_{R_{\leq[x]}}(y) = \mu_R(y, x)$ and $\nu_{R_{\leq[x]}}(y) = \nu_R(y, x)$.

$R_{\geq[x]}$ and $R_{\leq[x]}$ are called the dominating class of x and the class dominated by x, respectively.

Remark 1 *The notions of the dominating class of x and the class dominated by x are generalizations of the classical notions $\uparrow x$ and $\downarrow x$ in a usual poset.*

Next, we recall the definition of the upper bounds, the lower bounds, the supremum and the infimum on intuitionistic fuzzy ordered sets.

Definition 6 [24] *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X.*

(i) *The set of upper bounds of A with respect to R is the intuitionistic fuzzy subset of X defined by:*

$$U(R, A)(y) = \bigcap_{x\in A} R_{\geq[x]}(y), \ for\ any\ y \in X; \tag{3}$$

(ii) *The set of lower bounds of A with respect to R is the intuitionistic fuzzy subset of X defined by:*

$$L(R, A)(y) = \bigcap_{x\in A} R_{\leq[x]}(y), \ for\ any\ y \in X. \tag{4}$$

Definition 7 [24] *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X. An element $x \in X$ is called the least upper bound (or a supremum) of A with respect to R if:*

(i) *$x \in Supp(U(R, A))$ and*

(ii) *for all other $y \in Supp(U(R, A))$, $\mu_R(x, y) > 0$ or ($\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1$).*

An element $x \in X$ is called the greatest lower bound (or an infimum) of A with respect to R if:

(i) *$x \in Supp(L(R, A))$ and*

(ii) *for all other $y \in Supp(L(R, A))$, $\mu_R(y, x) > 0$ or ($\mu_R(y, x) = 0$ and $\nu_R(y, x) < 1$).*

Remark 2 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X. If the supremum and the infimum of A with respect to R exist, then from the perfect antisymmetry of R they are unique and denoted by $\sup_R(A)$ and $\inf_R(A)$, respectively.*

Definition 8 [24] *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) is called an intuitionistic fuzzy lattice with respect to the intuitionistic fuzzy order R (or simply, intuitionistic fuzzy lattice) if each pair of elements $\{x, y\}$ of X has a supremum and an infimum.*

Example 2 Let R be an intuitionistic fuzzy relation on $X = \{a, b, c, d, e\}$ defined by the following tables:

$\mu_R(\cdot,\cdot)$	a	b	c	d	e
a	1	0.7	0	0	0.1
b	0	1	0	0	0.1
c	0.5	0.7	1	1	0.8
d	0	0	0	1	0.5
e	0	0	0	0	1

$\nu_R(\cdot,\cdot)$	a	b	c	d	e
a	0	0.2	1	1	0.7
b	0.8	0	1	0.1	0.8
c	0.3	0.2	0	0	0.1
d	0.8	1	1	0	0.4
e	0.7	0.8	0.7	0.6	0

It is easy to verify that R is reflexive, perfect antisymmetric and min-max transitive. This implies that (X, μ_R, ν_R) is an intuitionistic fuzzy ordered set. The following table describes the supremum and the infimum of any subset of two elements $\{x, y\}$ of X.

$\{x, y\}$	$\sup_R\{x, y\}$	$\inf_R\{x, y\}$
$\{a, b\}$	b	a
$\{a, c\}$	a	c
$\{a, d\}$	d	a
$\{a, e\}$	e	a
$\{b, c\}$	b	c
$\{b, d\}$	e	a
$\{b, e\}$	e	b
$\{c, d\}$	d	c
$\{c, e\}$	e	c
$\{d, e\}$	e	d

So, (X, μ_R, ν_R) is an intuitionistic fuzzy lattice.

Definition 9 [24] *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set.*

(i) *An element $\top \in X$ is called the greatest element (the maximum) of X with respect to R or the intuitionistic fuzzy maximum of X if*

$$\begin{cases} \mu_R(x, \top) > 0, \\ \quad \text{or} \\ \mu_R(x, \top) = 0 \ \text{ and } \nu_R(x, \top) < 1 \end{cases}$$

for all $x \in X$.

(ii) *An element $\perp \in X$ is called the smallest element (the minimum) of X with respect to R or the intuitionistic fuzzy minimum if*

$$\begin{cases} \mu_R(\perp, x) > 0\,, \\ \quad \text{or} \\ \mu_R(\perp, x) = 0 \ \text{ and } \nu_R(\perp, x) < 1 \end{cases}$$

for all $x \in X$.

3.2. Basic Characterizations of Intuitionistic Fuzzy Complete Lattices

In this subsection, we introduce the notion of intuitionistic fuzzy complete lattice and show its basic characterizations.

Definition 10 *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) is called an intuitionistic fuzzy complete lattice if $\sup_R(A)$ and $\inf_R(A)$ exist for every nonempty subset $A \subseteq X$.*

Remark 3 *Every intuitionistic fuzzy complete lattice must have a greatest element (or a maximum) and a smallest element (or a minimum). The greatest element will be denoted by $\top_X$ and the smallest element $\perp_X$. It obviously holds that*

$$\top_X = \sup_R(X) = \inf_R(\emptyset) \text{ and } \perp_X = \inf_R(X) = \sup_R(\emptyset).$$

The following lemma is immediate.

Lemma 1 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set, A be a subset of X and $x \in X$. Then it holds that*

(i) $x = \sup_R(A)$ *with respect to R if and only if $x = \inf_{R^t}(A)$ with respect to R^t;*

(ii) $x = \inf_R(A)$ *with respect to R if and only if $x = \sup_{R^t}(A)$ with respect to R^t;*

(iii) (X, μ_R, ν_R) *is intuitionistic fuzzy complete lattice if and only if $(X, \mu_{R^t}, \nu_{R^t})$ is intuitionistic fuzzy complete lattice.*

The following results characterize the intuitionistic fuzzy complete lattices in terms of the existence of the supremum or the existence of the infimum of their subsets, and in terms of their intuitionistic fuzzy chains and maximal chains. These results are the main results of our conference paper [21], and will be used in the next section as auxiliary results to show that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa. We recall them here and adding their proofs since that the proofs were not mentioned in that conference paper [21].

Theorem 1 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set. The following statements hold:*

(i) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if $\sup_R(A)$ exists for all $A \subseteq X$;*

(ii) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if* $\inf_R(A)$ *exists for all* $A \subseteq X$.

Proof Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and $A \subseteq X$.

(i) Suppose that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, then it obviously holds that $\sup_R(A)$ exists for all $A \subseteq X$.

Conversely, suppose that $\sup_R(A)$ exists for all $A \subseteq X$ and we will show that every nonempty subset $A \subseteq X$ has an infimum. Let $L(R, A)$ be the intuitionistic fuzzy set of lower bounds of A with respect to R. Then, it holds that $\sup_R(Supp(L(R, A)))$ exists. We set $m = \sup_R(Supp(L(R, A)))$ and we show that $m = \inf_R(A)$.

First, since $m \in Supp(U(R, Supp(L(R, A))))$, then it holds that

$$\mu_{U(R,Supp(L(R,A)))}(x) > 0 \text{ or } [\mu_{U(R,Supp(L(R,A)))}(x) = 0 \text{ and } \nu_{U(R,Supp(L(R,A)))}(x) < 1].$$

By (3) it follows that $U(R, Supp(L(R, A)))(y) = \bigcap\limits_{x \in Supp(L(R,A))} R_{\geq[x]}(y)$.

Since

$$R_{\geq[x]} = \{\langle y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)\rangle \ / \ y \in X\},$$

where

$$\mu_{R_{\geq[x]}}(y) = \mu_R(x, y) \text{ and} \nu_{R_{\geq[x]}}(y) = \nu_R(x, y)$$

and by the fact that $R_{\geq[x]} = R^t_{\leq[x]}$ and $U(R, A) = L(R^t, A)$, it follows that

$$U(R, Supp(L(R, A))) = L(R^t, Supp(L(R, A))) = L(R, A).$$

Hence, $m \in Supp(L(R, A))$.

In the same way, for all $y \in Supp(L(R, A))$, it holds that $\mu_R(y, m) > 0$ or $(\mu_R(y, m) = 0$ and $\nu_R(y, m) < 1)$. Thus $m = \inf_R(A)$, which implies that $\inf_R(A)$ exists. Therefore, (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

(ii) Follows from Lemma 1 and (i).

Remark 4 *In the above Theorem 1 the existence of* $\inf_R(\emptyset)$ *guarantees the existence of the greatest element of* (X, μ_R, ν_R)*, and in similar way, the existence of* $\sup_R(\emptyset)$ *guarantees the smallest element of* (X, μ_R, ν_R)*. So an equivalent formulation of Theorem 1 will be*

(i) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if it has the smallest element and* $\sup_R(A)$ *exists for all nonempty* $A \subseteq X$*;*

(ii) (X, μ_R, ν_R) *is an intuitionistic fuzzy complete lattice if and only if it has the greatest element and* $\inf_R(A)$ *exists for all nonempty* $A \subseteq X$.

Theorem 2 *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy lattice. Then, the following statements are equivalent:*

(i) (X, μ_R, ν_R) *is intuitionistic fuzzy complete lattice;*

(ii) *(X, μ_R, ν_R) is intuitionistic fuzzy chain-complete (i.e., every nonempty intuitionistic fuzzy chain in (X, μ_R, ν_R) has a supremum and an infimum);*

(iii) *Every maximal intuitionistic fuzzy chain of X is an intuitionistic fuzzy complete lattice.*

Proof (i)⇒ (ii) is obvious.

To prove (ii)⇒ (iii), let C be a maximal intuitionistic fuzzy chain (with respect to set inclusion) of X. First, we will show that C has an intuitionistic fuzzy maximum and an intuitionistic fuzzy minimum. Since C is an intuitionistic fuzzy chain in (X, μ_R, ν_R), it holds from (ii) that C has a supremum and an infimum. By using the fact that C is maximal (with respect to set inclusion) we obtain that $c_1 = \sup_R(C)$ is the maximum and $c_2 = \inf_R(C)$ is the minimum.

Second, let $A \subseteq C$. Then it holds that A is an intuitionistic fuzzy chain. From (ii), it follows that $\sup_R(A)$ exists in (X, μ_R, ν_R) and we denote it by m. Now, it suffices to show that $m \in C$.

Suppose that $m \notin C$, it follows three cases:

(a) If $[\mu_R(x, m) > 0$ or $(\mu_R(x, m) = 0$ and $\nu_R(x, m) < 1)]$ or $[\mu_R(m, x) > 0$ or $(\mu_R(m, x) = 0$ and $\nu_R(m, x) < 1)]$ for all $x \in C$, then $C \cup \{m\}$ is an intuitionistic fuzzy chain in (X, μ_R, ν_R). This is a contradiction with the fact that C is maximal.

(b) If there exists $x \in C$ such that $[\mu_R(x, m) = 0$ and $\nu_R(x, m) = 1]$, then it follows from the transitivity of R that

$$\mu_R(x, c_1) \wedge \mu_R(c_1, m) \leq \mu_R(x, m), \text{ and } \nu_R(x, c_1) \vee \nu_R(c_1, m) \geq \nu_R(x, m).$$

Since $[\mu_R(x, m) = 0$ and $\nu_R(x, m) = 1]$, then it holds that $\mu_R(c_1, m) = 0$ and $\nu_R(c_1, m) = 1$. Hence, $\sup_R\{c_1, m\} \notin C$. Thus, $C \bigcup \{\sup_R\{c_1, m\}\}$ is an intuitionistic fuzzy chain, which is a contradiction with the maximality of C.

(c) If there exists $x \in C$ such that $[\mu_R(m, x) = 0$ and $\nu_R(m, x) = 1]$, then it follows similarly as (b).

As a consequence of the above cases we get $m \in C$. Thus, A has a supremum in C. Therefore, C is an intuitionistic fuzzy complete lattice follows from Theorem 1.

(iii)⇒ (i) Suppose that every maximal intuitionistic fuzzy chain of X is an intuitionistic fuzzy complete lattice. We will show that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

Let $A \subseteq X$ and $\mathcal{BFC}(Supp(U(R, A)))$ denote the set of all intuitionistic fuzzy chains $C \subseteq Supp(U(R, A))$. Ordered in the classical way by $C_1 \sqsubseteq C_2$ if and only if C_1 is an intuitionistic fuzzy filter of C_2, i.e., $C_1 \subseteq C_2$ or [if $x \in C_1$ and $y \in C_2$ with $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$, then $y \in C_1$].

Next, let $\{C_i : i \in I \subseteq N\}$ be a chain of $\mathcal{BFC}(Supp(U(R, A)))$ under the crisp order defined above. On the one hand, since C_i is an intuitionistic fuzzy chain of $Supp(U(R, A))$ and $C_i \subseteq C_{i+1}$ for all $i \in I$, then $\bigcup_{i \in I} C_i$ is an intuitionistic fuzzy chain of $Supp(U(R, A))$. Hence, $\bigcup_{i \in I} C_i \in \mathcal{BFC}(Supp(U(R, A)))$. On the other hand, $\bigcup_{i \in I} C_i$ is an upper bound of $\{C_i\}_{i \in I}$.

By Zorn's Lemma, we know that $\mathcal{BFC}(Supp(U(R, A)))$ has a maximal element denoted by C_m with respect to the above crisp order $\sqsubseteq$.

Let K be a maximal intuitionistic fuzzy chain such that $C_m \subseteq K$. By hypothesis, K is an intuitionistic fuzzy complete lattice, which implies that C_m has an infimum denoted by c in (K, μ_R, ν_R).

Now, we will show that $c = \sup_R(A)$. Indeed, let $x \in A$. Since $C_m \subseteq Supp(U(R, A))$, then it holds that

$$\mu_R(x, y) > 0 \text{ or } (\mu_R(x, y) = 0 \text{ and } \nu_R(x, y) < 1) \text{ for all } y \in C_m\,.$$

Hence,

$$\mu_R(x, c) > 0 \text{ or } (\mu_R(x, c) = 0 \text{ and } \nu_R(x, c) < 1)\,.$$

Thus, $c \in Supp(U(R, A))$. For all other $y \in Supp(U(R, A))$, it holds that

$$\mu_R(c, y) > 0 \text{ or } (\mu_R(c, y) = 0 \text{ and } \nu_R(c, y) < 1)\,.$$

Otherwise, we get a contradiction with the maximality of C_m. Thus, $c = \sup_R(A)$. Now, (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice follows from Theorem 1 (i).

4. Fixed Point Property for Intuitionistic Fuzzy Lattices

In this section, we show that any intuitionistic fuzzy complete lattice has the fixed point property and vice versa, i.e., an intuitionistic fuzzification of Tarski-Davis's fixed point theorem.

4.1. An Intuitionistic Fuzzification of Tarski's Fixed Point Theorem

In this subsection, we show that the intuitionistic fuzzy complete lattices have the fixed point property. Moreover, we show that the set of fixed points of an intuitionistic fuzzy monotone mapping of an intuitionistic fuzzy complete lattice into itself is also an intuitionistic fuzzy complete lattice. First, we recall the following definitions.

Definition 11 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set. A mapping $f : X \to X$ is called monotone mapping with respect to the intuitionistic fuzzy order R or (intuitionistic fuzzy monotone, for short) if $\mu_R(f(x), f(y)) \geq \mu_R(x, y)$ and $\nu_R(f(x), f(y)) \leq \nu_R(x, y)$, for all $x, y \in X$.*

Definition 12 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set. Then*

(i) *an element $x \in X$ is called a fixed point of a mapping $f : X \to X$ if $f(x) = x$. The set of all fixed points of f will be denoted by $Fix(f)$.*

(ii) *X is said to have the fixed point property with respect to the intuitionistic fuzzy order R, or (the fixed point property, for short) if every intuitionistic fuzzy monotone mapping f of (X, μ_R, ν_R) into itself has a fixed point.*

The following theorem shows that any intuitionistic fuzzy complete lattice has the fixed point property, i.e., Tarski's fixed point theorem for intuitionistic fuzzy complete lattices.

Theorem 3 *Any intuitionistic fuzzy complete lattice has the fixed point property.*

Proof Setting that

$$A = \{x \in X \mid \mu_R(x, f(x)) > 0 \text{ or } (\mu_R(x, f(x)) = 0 \text{ and } \nu_R(x, f(x)) < 1)\}.$$

Since $\perp \in A$, then $A \neq \emptyset$. Using the fact that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice, we obtain that A has a supremum, denoted by m. We will show that m is a fixed point of f. By the intuitionistic fuzzy monotonicity of f, we obtain for $x \in A$ that

$$\mu_R(f(x), f(f(x))) \geq \mu_R(x, f(x)) > 0,$$

or

$$(\mu_R(f(x), f(f(x))) = 0 \text{ and } \nu_R(f(x), f(f(x))) \leq \nu_R(x, f(x)) < 1),$$

this implies that

$$f(A) \subseteq A. \tag{5}$$

We also get that

$$\mu_R(f(x), f(m)) \geq \mu_R(x, m) > 0 \tag{6}$$

or

$$\mu_R(f(x), f(m)) = 0 \text{ and } \nu_R(f(x), f(m)) \leq \nu_R(x, m) < 1. \tag{7}$$

By (6) and (7), it holds that $f(m) \in Supp(U(R, A))$. Now, since $m = \sup_R(A)$ and $f(m) \in Supp(U(R, A))$, it follows that

$$\mu_R(m, f(m)) > 0 \tag{8}$$

or

$$\mu_R(m, f(m)) = 0 \text{ and } \nu_R(m, f(m)) \leq \nu_R(x, m) < 1. \tag{9}$$

By (8) and (9) we get that $m \in A$, which implies from (5) that $f(m) \in A$. Since $m = \sup_R(A)$ and $f(m) \in A$, then it follows that

$$\mu_R(f(m), m) > 0 \tag{10}$$

or

$$\mu_R(f(m), m) = 0 \text{ and } \nu_R(f(m), m) < 1. \tag{11}$$

Therefore, $m = f(m)$ follows from (8)·(9)·(10)·(11) and the perfect antisymmetry of R.

In the same direction, the following theorem shows that the set of fixed points of an intuitionistic fuzzy monotone mapping of an intuitionistic fuzzy complete lattice into itself is also an intuitionistic fuzzy complete lattice. First, we need to show the following lemma.

Lemma 2 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice, $f : X \to X$ be an intuitionistic fuzzy monotone mapping and $A \subseteq Fix(f)$. Then it holds that*

(i) *If* $B = \{x \in X \mid ((\mu_R(y, f(x)) > 0 \text{ or } (\mu_R(y, f(x)) = 0 \text{ and } \nu_R(y, f(x)) < 1)) \text{ and } (\mu_R(f(x), x) > 0 \text{ or } (\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) < 1)), \forall y \in A)\}$, *then* $\inf_R(B)$ *is a fixed point of* f *in* B.

(ii) *If* $C = \{x \in X \mid ((\mu_R(f(x), y) > 0 \text{ or } (\mu_R(f(x), y) = 0 \text{ and } \nu_R(f(x), y) < 1)) \text{ and } (\mu_R(x, f(x)) > 0 \text{ or } (\mu_R(x, f(x)) = 0 \text{ and } \nu_R(x, f(x)) < 1)), \forall y \in A)\}$, *then* $\sup_R(C)$ *is a fixed point of* f *in* C.

Proof Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice, $f : X \to X$ be an intuitionistic fuzzy monotone mapping and $A \subseteq Fix(f)$. Then
(i) Let $m = \inf_R(B)$, i.e.,
(a) $m \in Supp(L(R, B))$ and
(b) for all other $y \in Supp(L(R, B))$, $\mu_R(y, m) > 0$ or ($\mu_R(y, m) = 0$ and $\nu_R(y, m) < 1$).
Hence,

$$\mu_R(m, x) > 0 \text{ or } (\mu_R(m, x) = 0 \text{ and } \nu_R(m, x) < 1)$$

for all $x \in B$. The monotonicity of f implies that

$$\mu_R(f(m), f(x)) > 0 \text{ or } (\mu_R(f(m), f(x)) = 0 \text{ and } \nu_R(f(m), f(x)) < 1)$$

for all $x \in B$, since $\mu_R(f(x), x) > 0$ or ($\mu_R(f(x), x) = 0$ and $\nu_R(f(x), x) < 1$) for all $x \in B$, it follows from the transitivity of R that

$$\mu_R(f(m), x) > 0 \text{ or } (\mu_R(f(m), x) = 0 \text{ and } \nu_R(f(m), x) < 1)\,.$$

This implies that $f(m) \in Supp(L(R, B))$. Hence,

$$\mu_R(f(m), m) > 0 \text{ or } (\mu_R(f(m), m) = 0 \text{ and } \nu_R(f(m), m) < 1)\,. \tag{12}$$

In the other hand, since

$$(\mu_R(y, f(x)) > 0 \text{ or } (\mu_R(y, f(x)) = 0 \text{ and } \nu_R(y, f(x)) < 1))$$

and

$$\mu_R(f(x), x) > 0 \text{ or } (\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) < 1)$$

for all $y \in A$ and $x \in B$, it follows from the transitivity of R that

$$\mu_R(y, x) > 0 \text{ or } (\mu_R(y, x) = 0 \text{ and } \nu_R(y, x) < 1)$$

for all $y \in A$. Hence,

$$\mu_R(y, m) > 0 \text{ or } (\mu_R(y, m) = 0 \text{ and } \nu_R(y, m) < 1)$$

for all $y \in A$. The monotonicity of f implies that $\mu_R(f(f(y)), f(f(m))) > 0$ or $(\mu_R(f(f(y)), f(f(m))) = 0$ and $\nu_R(f(f(y)), f(f(m))) < 1)$, for all $y \in A$. Since $A \subseteq Fix(f)$ it holds that

$$\mu_R(y, f(f(m))) > 0 \text{ or } (\mu_R(y, f(f(m))) = 0 \text{ and } \nu_R(y, f(f(m))) < 1)\,. \tag{13}$$

Also, Eq. (12) and the monotonicity of f imply that

$$\mu_R(f(f(m)), f(m)) > 0 \text{ or } (\mu_R(f(f(m)), f(m)) = 0 \text{ and } \nu_R(f(f(m)), f(m)) < 1)\,. \tag{14}$$

Eqs. (13) and (14) imply that $f(m) \in B$. Thus,

$$\mu_R(m, f(m)) > 0 \text{ or } (\mu_R(m, f(m)) = 0 \text{ and } \nu_R(m, f(m)) < 1)\,. \tag{15}$$

Thus, Eqs. (12), (15) and the perfect antisymmetry of R imply that $m = f(m)$. Therefore, $m = \inf_R(B)$ is a fixed point of f in B.

(ii) Follows from Lemma 1 and (i).

Theorem 4 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy complete lattice and $f : X \to X$ be an intuitionistic fuzzy monotone mapping. Then the set $Fix(f)$ is an intuitionistic fuzzy complete lattice.*

Proof Suppose that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice and let $f : X \to X$ be an intuitionistic fuzzy monotone mapping. Theorem 3 guarantees that $Fix(f)$ is a nonempty set. Now, let A be a subset of $Fix(f)$. We will show that A has a supremum with respect to R in $Fix(f)$, and then Theorem 1 (i) implies that $(Fix(f), \mu_R, \nu_R)$ is an intuitionistic fuzzy complete lattice.

Let $B = \{x \in X \mid (\forall y \in A)(\mu_R(y, f(x)) > 0$ or $(\mu_R(y, f(x)) = 0$ and $\nu_R(y, f(x)) < 1))$, and $(\mu_R(f(x), x) > 0$ or $(\mu_R(f(x), x) = 0$ and $\nu_R(f(x), x) < 1))\}$. Since (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice then it holds that B has an infimum, denoted by m. Lemma 2 guarantees that m is a fixed point of f in B, i.e., $m = f(m)$ and $m \in B$.

Next, we will show that m is the supremum of A with respect to R.

Since $m = f(m) \in B$ and $(\mu_R(y, x) > 0$ or $(\mu_R(y, x) = 0$ and $\nu_R(y, x) < 1))$ for all $y \in A$ and $x \in B$, it follows that

$$\mu_R(y, m) > 0 \text{ or } (\mu_R(y, m) = 0 \text{ and } \nu_R(y, m) < 1)$$

for all $y \in A$. Hence,

$$m \in Supp(U(R,A))\,. \tag{16}$$

Now, let s be an other fixed point of f and $s \in Supp(U(R,A))$. Then it holds that

$$\mu_R(y,s) > 0 \text{ or } (\mu_R(y,s) = 0 \text{ and } \nu_R(y,s) < 1)$$

for all $y \in A$.

Since $A \subseteq Fix(f)$, it follows from the monotonicity of f that

$$\mu_R(y, f(s)) > 0 \text{ or } (\mu_R(y, f(s)) = 0 \text{ and } \nu_R(y, f(s)) < 1)$$

for all $y \in A$, Again, since $f(s) = s$ which implies due to the reflexivity of R that $\mu_R(f(s), s) = 1 > 0$. Hence, $s \in B$. Thus,

$$\mu_R(m,s) > 0 \text{ or } (\mu_R(m,s) = 0 \text{ and } \nu_R(m,s) < 1) \tag{17}$$

for all $s \in Supp(U(R,A)) \cap Fix(f)$. Therefore, Eqs. (16) and (17) imply that m is the supremum of A with respect to R in $Fix(f)$.

Combining Theorem 3 and Theorem 4, we obtain the following characterizations of intuitionistic fuzzy complete lattice in terms of least and greatest fixed points of its intuitionistic fuzzy monotone maps.

Corollary 1 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. Then, the following s-tatements are equivalent:*

(i) *(X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice;*

(ii) *Every intuitionistic fuzzy monotone mapping $f : X \to X$ has a least fixed point;*

(iii) *Every intuitionistic fuzzy monotone mapping $f : X \to X$ has a greatest fixed point.*

4.2. An Intuitionistic Fuzzification of Davis's Fixed Point Theorem

In this subsection, we first introduce the notions of intuitionistic fuzzy ascending (resp. descending) chain and using them to establish a characterization theorem of non-complete intuitionistic fuzzy lattice. Based on this characterization, we show that any intuitionistic fuzzy lattice has the fixed point property is complete, i.e., an intuitionistic fuzzification of Davis's fixed point theorem.

Definition 13 *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and $\{a_i\}_{i \in I \subseteq N}$ be a subset of elements of X. Then*

(i) *$\{a_i\}_{i \in I \subseteq N}$ is called an intuitionistic fuzzy ascending chain (or an ascending chain with respect to R) if $\mu_R(a_i, a_{i+1}) > 0$ or $(\mu_R(a_i, a_{i+1}) = 0$ and $\nu_R(a_i, a_{i+1}) < 1)$, for all $i \in I$. Descending intuitionistic fuzzy chains (or an descending chain with respect to R) are defined dually.*

(ii) (X, μ_R, ν_R) *is said to satisfy the intuitionistic fuzzy ascending chain condition (or the* ACC_R*, for short) if every intuitionistic fuzzy ascending chain* $\{a_i\}_{i\in I\subseteq N}$ *of elements of* X *is eventually stationary (i.e., there exist a positive integer* $n \in I$ *such that* $a_m = a_n$ *for all* $m > n$*). In other words,* (X, R) *contains no infinite intuitionistic fuzzy ascending chain.*

(iii) *Similarly,* (X, μ_R, ν_R) *is said to satisfy the intuitionistic fuzzy descending chain condition (or the* DCC_R*, for short) if every intuitionistic fuzzy descending chain* $\{a_i\}_{i\in I\subseteq N}$ *of elements of* X *is eventually stationary.*

The following result establish a characterization that an intuitionistic fuzzy lattice isn't complete in terms of its intuitionistic fuzzy ascending (resp. descending) chains.

Theorem 5 *Let* (X, μ_R, ν_R) *be an intuitionistic fuzzy lattice.* (X, μ_R, ν_R) *is not an intuitionistic fuzzy complete lattice if and only if there exists an intuitionistic fuzzy chain* C *satisfying the* ACC_R *and having no infimum and there exists an intuitionistic fuzzy chain* D *satisfying the* DCC_R *and having no supremum such that:*

(i) $\mu_R(d, c) > 0$ *or* $(\mu_R(d, c) = 0$ *and* $\nu_R(d, c) < 1)$*, for any* $d \in D$ *and* $c \in C$ *;*

(ii) *For all* $x \in X$*, either there exists* $c \in C$ *with* $(\mu_R(x, c) = 0$ *and* $\nu_R(x, c) = 1)$ *or there exists* $d \in D$ *with* $(\mu_R(d, x) = 0$ *and* $\nu_R(d, x) = 1)$*, i.e., there doesn't exist an element* $x \in X$ *such that*

$$x \in Supp(L(R, C)) \bigcap Supp(U(R, D)) .$$

Proof Let (X, μ_R, ν_R) be an intuitionistic fuzzy lattice. Obviously that if there exists on (X, μ_R, ν_R) an intuitionistic fuzzy chain C satisfying the ACC_R and having no infimum or there exists an intuitionistic fuzzy chain D satisfying the DCC_R and having no supremum, then it is not an intuitionistic fuzzy complete lattice. Conversely, assume that (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice. Then either (X, μ_R, ν_R) has no greatest element $\top_X$ or there exists an intuitionistic fuzzy chain $C \subseteq X$ that satisfies the ACC_R and has no infimum. Suppose that (X, r) has greatest element $\top_X$ and that every intuitionistic fuzzy chain C in X satisfying the ACC_R has an infimum. We will show that every subset A of X has a supremum, which proves by applying Theorem 1 that (X, μ_R, ν_R) is an intuitionistic fuzzy complete lattice.

Indeed, let A be a subset of X and $U(R, A)$ the set of upper bounds of A with respect to R. Since $U(R, A)(\top_X) = 1$ (i.e., $\top_X \in Supp(U(R, A))$), it follows that

$$Supp(U(R, A)) \neq \emptyset.$$

Now, let $IFC_{ACC}(U(R, A))$ denote the set of all intuitionistic fuzzy chains C containing in $Supp(U(R, A))$ and satisfying the ACC_R. Ordered in the classical way by $C_1 \sqsubseteq C_2$ if and only if C_1 is an intuitionistic fuzzy filter of C_2 with respect to R, i.e., $C_1 \subseteq C_2$ and $(\forall x \in C_1, \forall y \in C_2)$ $(\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1) \Rightarrow y \in C_1)$.

Next, let $\{C_i \mid i \in I \subseteq N\}$ be a chain of $IFC_{ACC}(U(R,A))$ under the order defined above. On the one hand, since C_i is an intuitionistic fuzzy chain of $Supp(U(R,A))$ satisfying the ACC_R and $C_i \subseteq C_{i+1}$ for all $i \in I$, then it follows that $\bigcup_{i\in I} C_i$ is an intuitionistic fuzzy chain of $Supp(U(R,A))$ satisfying the ACC_R. Hence,

$$\bigcup_{i\in I} C_i \in IFC_{ACC}(U(R,A))\,.$$

Now, we will show that $C_j \sqsubseteq \bigcup_{i\in I} C_i$, for any $j \in I$. Indeed, let $x \in C_j$ and $y \in \bigcup_{i\in I} C_i$. Since $y \in \bigcup_{i\in I} C_i$, it follows that there exists $k \in I$ such that $y \in C_k$. We consider two cases:

(i) If $j \leq k$, since $C_j \sqsubseteq C_k$, it follows that $y \in C_j$. Hence, C_j is an intuitionistic fuzzy filter of $\bigcup_{i\in I} C_i$ with respect to R.

(ii) If $j > k$, then it trivially holds that $y \in C_j$. Hence, C_j is an intuitionistic fuzzy filter of $\bigcup_{i\in I} C_i$ with respect to R.

Thus, $C_j \sqsubseteq \bigcup_{i\in I} C_i$, for any $j \in I$. Therefore, $\bigcup_{i\in I} C_i$ is an upper bound of $\{C_j\}_{j\in I}$. By using Zorn's Lemma, it follows that $IFC_{ACC}(U(R,A))$ has a maximal element denoted by C_m. By hypothesis, C_m has an infimum in (X, μ_R, ν_R). Setting $c = \inf_R(C_m)$, and we will prove that $c = \sup_R(A)$. Indeed, since $C_m \subseteq Supp(U(R,A))$, it follows that $\mu_{U(R,A)}(s) > 0$ or ($\mu_{U(R,A)}(s) = 0$ and $\nu_{U(R,A)}(s) < 1$), for all $s \in C_m$. Since $U(R,A)(s) = \bigcap\limits_{x\in A} R_{\geq[x]}(s)$, it follows that

$$\mu_R(x,s) > 0 \text{ or } (\mu_R(x,s) = 0 \text{ and } \nu_R(x,s) < 1)$$

for all $x \in A$ and $s \in C_m$. This implies that

$$\mu_R(x,c) > 0 \text{ or } (\mu_R(x,c) = 0 \text{ and } \nu_R(x,c) < 1)$$

for all $x \in A$. Hence,

$$U(R,A)(c) = \bigcap_{x\in A} R_{\geq[x]}(c) > 0\,,$$

i.e.,

$$\mu_{U(R,A)}(c) > 0 \text{ or } (\mu_{U(R,A)}(c) = 0 \text{ and } \nu_{U(R,A)}(c) < 1)\,.$$

Thus, $c \in Supp(U(R,A))$.

On the other hand, suppose that $\mu_R(c,y) = 0$ and $\nu_R(c,y) = 1$ for some $y \in Supp(U(R,A))$. Since $\inf_R\{c,y\} \in Supp(U(R,A))$ and

$$[\mu_R(\inf_R\{c,y\},x) > 0 \text{ or } (\mu_R(\inf_R\{c,y\},x) = 0 \text{ and } \nu_R(\inf_R\{c,y\},x) < 1)]$$

for all $x \in C_m$, it follows that $C_m \cup \inf_R\{c,y\}$ is an intuitionistic fuzzy chain of $Supp(U(R,A))$. Also, it obviously holds that $C_m \cup \inf_R\{c,y\}$ satisfies the ACC_R.

Hence, $C_m \cup \inf_R\{c, y\} \in IFC_{ACC}(U(R, A))$, a contradiction with the maximality of C_m. Thus, for all other $y \in Supp(U(R, A))$,

$$\mu_R(c, y) > 0 \text{ or } (\mu_R(c, y) = 0 \text{ and } \nu_R(c, y) < 1).$$

Therefore, $c = \sup_R(A)$. We conclude that if (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice, then either (X, μ_R, ν_R) has no greatest element or there exists an intuitionistic fuzzy chain $C \subseteq X$ that satisfies the ACC_R and has no infimum. Also, if (X, μ_R, ν_R) has no greatest element, taking $C = C_m$ an intuitionistic fuzzy chain satisfying the ACC_R but having no infimum; otherwise we take $C = \emptyset$ (because $\emptyset$ can be considered as an intuitionistic fuzzy chain satisfying the ACC_R and since (X, μ_R, ν_R) has no greatest element, it follows that $\emptyset$ having no infimum).

Dually, let $IFC_{DCC}(L(R, C))$ denote the set of all intuitionistic fuzzy chains $D \subseteq Supp(L(R, C))$ satisfying the DCC_R. Ordered in the classical way by $D_1 \sqsubseteq D_2$ if and only if D_1 is an intuitionistic fuzzy ideal of D_2 with respect to R, i.e., $D_1 \subseteq D_2$ and $\forall x \in D_1$ and $y \in D_2$,

$$(\mu_R(y, x) > 0 \text{ or } (\mu_R(y, x) = 0 \text{ and } \nu_R(y, x) < 1) \Rightarrow y \in D_1).$$

If $L(R, C) \neq \emptyset$, then $IFC_{DCC}(L(R, C)) \neq \emptyset$. By using Lemma 1 and the same steps as above, we will obtain that $IFC_{DCC}(L(R, C))$ has a maximal element denoted by D_m. We take $D = D_m$ if $IFC_{DCC}(L(R, A)) \neq \emptyset$ and $D = \emptyset$, otherwise.

Moreover,

(i) Since $D \subseteq Supp(L(R, C))$, it follows that $\mu_R(d, c) > 0$ or $(\mu_R(d, c) = 0$ and $\nu_R(d, c) < 1)$ for any $d \in D$ and $c \in C$.

(ii) Let $x \in X$. Suppose that $\mu_R(x, c) > 0$ or $(\mu_R(x, c) = 0$ and $\nu_R(x, c) < 1)$, for all $c \in C$ and $\mu_R(d, x) > 0$ or $(\mu_R(d, x) = 0$ and $\nu_R(d, x) < 1)$, for all $d \in D$. Since $x \in Supp(L(R, C))$, it follows that $Supp(L(R, C)) \neq \emptyset$. Hence, $FC_{DCC}(L(R, C)) \neq \emptyset$. Thus, $D = D_m$. On the other hand, since $x \in Supp(L(R, C))$ and $\inf_R(C)$ doesn't exist, it follows that there exists $y \in Supp(L(R, C))$ such that $\mu_R(y, x) = 0$ and $\nu_R(y, x) = 1$. Since

$$\mu_R(d, x) > 0 \text{ or } (\mu_R(d, x) = 0 \text{ and } \nu_R(d, x) < 1)$$

for all $d \in D$ and

$$\mu_R(x, \sup_R\{x, y\}) > 0 \text{ or } (\mu_R(x, \sup_R\{x, y\}) = 0 \text{ and } \nu_R(x, \sup_R\{x, y\}) < 1),$$

it follows from the transitivity of R that

$$\mu_R(d, \sup_R\{x, y\}) > 0 \text{ or } (\mu_R(d, \sup_R\{x, y\}) = 0 \text{ and } \nu_R(d, \sup_R\{x, y\}) < 1)$$

for all $d \in D$. Hence, $D \cup \{\sup_R\{x, y\}\}$ is an intuitionistic chain of $Supp(L(R, C))$. Also, since D satisfying the DCC_R, it is obvious that $D \cup \{\sup_R\{x, y\}\}$ satisfying the

DCC_R. Hence, $D \cup \{\sup_R\{x, y\}\} \in IFC_{DCC}(L(R, C))$. This is a contradiction with the maximality of D. Therefore, for all $x \in X$, either there exists $c \in C$ with

$$(\mu_R(x, c) = 0 \text{ and } \nu_R(x, c) = 1),$$

or there exists $d \in D$ with

$$(\mu_R(d, x) = 0 \text{ and } \nu_R(d, x) = 1)\,,$$

i.e., there does not exist an element $x \in X$ such that

$$x \in Supp(L(R, C)) \bigcap Supp(U(R, D)).$$

The following theorem shows that any intuitionistic fuzzy lattice has the fixed point property is complete.

Theorem 6 *Any intuitionistic fuzzy lattice has the fixed point property is complete.*

Proof Assume that (X, μ_R, ν_R) is not an intuitionistic fuzzy complete lattice and we will show that there exist an intuitionistic fuzzy monotone mapping $f : X \to X$ that has no fixed point. From Theorem 5 we know that there exists an intuitionistic fuzzy chain C satisfying the ACC_R and having no infimum and an intuitionistic fuzzy chain D satisfying the DCC_R and having no supremum. For any $x \in X$ we have that

$$C_x = \{c \in C \ : \ \mu_R(x, c) = 0 \text{ and } \nu_R(x, c) = 1\}$$

and

$$D_x = \{d \in D \ : \ \mu_R(d, x) = 0 \text{ and } \nu_R(d, x) = 1\}.$$

From Theorem 5 (ii) it is easy to see that for any $x \in X$, only one of the above two subsets is nonempty. Suppose that $C_x \neq \emptyset$. Since $C_x \subseteq C$ and C satisfies the ACC_R, then it follows that C_x has the greatest element.

Similarly, if we suppose that $D_x \neq \emptyset$, then from the fact that $D_x \subseteq D$ and D satisfies the DCC_R we get that D_x has the smallest element. Now, we define the mapping $f : X \to X$ as: $f(x)$ is the greatest element of C_x if $C_x \neq \emptyset$ or $f(x)$ is the smallest element of D_x, if $D_x \neq \emptyset$. Obviously, f is an intuitionistic fuzzy monotone mapping with respect to R. Next, we will show that f has no fixed point. Indeed, let $x \in X$. Since $f(x) \in C_x$ or $f(x) \in D_x$, it follows that

$$\mu_R(x, f(x)) = 0 \text{ and } \nu_R(x, f(x)) = 1\,,$$

or

$$\mu_R(f(x), x) = 0 \text{ and } \nu_R(f(x), x) = 1.$$

Thus, $x \neq f(x)$. Therefore, f has no fixed point.

5. Conclusion

In this paper, we have introduced the notion of intuitionistic fuzzy complete lattice and some characterizations have been expressed in terms of the supremum and the infimum of its subsets, chains and maximal chains. In the main contribution, we have shown that an intuitionistic fuzzy lattice is complete if and only if it satisfies the fixed point property, which makes the fixed point problem have a complete solution when we restrict to the class of intuitionistic fuzzy lattices. In future work, we will investigate other classes of intuitionistic fuzzy ordered sets satisfying the fixed point property, in order to combine the properties of these classes with those of the intuitionistic fuzzy monotone maps.

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