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THEORETICAL AND NUMERICAL RESULT FOR LINEAR SEMIDEFINITE PROGRAMMING BASED ON A NEW KERNEL FUNCTION

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Abstract. Kernel functions serve the central goal of creating new search directions for the primal-dual interior-point algorithm to solve linear optimization problems. A significantly improved primal-dual interior-point algorithm for linear optimization is presented based on a novel kernel function. We show a primal-dual interior-point technique for linear optimization based on a class of kernel functions that are eligible. This research presents a new efficient kernel function-based primal-dual IPM algorithm for semidefinite programming problems based on the Nesterov-Todd (NT) direction. With a new and simple technique, we propose a new kernel function to obtain an optimal solution of the perturbed problem (SDP)μ. We obtain the best-known complexity results,for small- and large-update namely $\mathscr{O}\left(p^{\frac{p+1}{2p}}\sqrt{n}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$ and $\mathscr{O}\left((pn)^{\frac{p+1}{2p}}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$ large update To prove the effectiveness of our proposed kernel function, we compare our numerical results with some alternatives presented by Touil et al. (2017).

Keywords: linear semidefinite programming; central trajectory methods; primal-dual interior point methods; kernel function.

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1. Introduction

Semi-defined programming extends linear programming where vector variables are changed to matrix variables. In particular, an optimization problem is defined as a semi-defined optimization problem (SDO) in its primal form.

$$(P) \qquad \min\left\{C \bullet X,\ A_i \bullet X = b_i,\ 1 \leq i \leq m,\ X \succeq 0\right\},$$

and its dual problems.

$$(D) \qquad \max\left\{b^T y:\ \sum_{i=1}^{m} y_i A_i + S = C,\ S \succeq 0\right\},$$

where $C,\ A_i \in \mathbf{S}^n,\ 1 \leq i \leq m,\ b = (b_1, b_2, ..., b_m) \in \mathbf{R}^m,\ y \in \mathbf{R}^m$. Here $\mathbf{S}^n$ denotes the space of $n \times n$ real symmetric matrices. In addition, $X \succeq 0$ indicates that X is a symmetric positive semidefinite matrix. The operator $\bullet$ denotes the standard inner product in $\mathbf{S}^n$, i.e., $C \bullet X = \mathrm{tr}(CX) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{n} C_{ij}X_{ij}$, in which tr represents the trace. Moreover, the matrices A_i are linearly independent.

Semidefinite programs are well known for their applications to different optimization problems. It is popular and widely used in mathematical programming and other fields, such as engineering and science, such as control theory, electronic structure problems, and statistics. [1,2]. Many IPMs for LO (e.g., [3,4,5,6,7]) are extended to SDO with great success due to their polynomial complexity and practical efficiency. For a detailed review, the reader is directed to [2,8,9].

The IPM relies on the barrier functions, which are defined by a large class of univariate functions, popularly referred to as the eligible kernel functions. [4], which has recently been efficiently utilized to construct new primal-dual IPMs for various optimization problems. Compared to the logarithmic kernel function, using certain kernel functions reduces the difference in complexity between methods with large updates and methods with small updates.

This paper aims is to create a new class of IPMs for SDO with large and small updates based on the following parametric kernel function: a two-barrier term.

$$\psi(t) = (p+2)t^2 - (p+3)t - \log(t) + \frac{1}{t^p}, \quad p \geq 2 \tag{1}$$

where p is a parameter.

We show that the iteration bounds are $\mathscr{O}\left((np)^{\frac{p+1}{2p}}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$ and $\mathscr{O}\left(p^{\frac{p+1}{2p}}\sqrt{n}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$ which are at least as good as the current best known bounds, which in some situations are at least as good as the current best known bounds, so far, $\mathscr{O}\left(\sqrt{n}\log(n)\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$. For small-update methods, the iteration bounds are $\mathscr{O}(\sqrt{n}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon})$, which are currently the best-known bounds. Based on the proposed kernel function, we show that the worst-case iteration complexity for primal-dual IPMS is $\mathscr{O}\left(n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right)$, which improves on the complexity achieved by El Ghami et al. in [3].

The structure of that paper is as follows: In Section 2, we start by going over the basics of IPMs for SDO, like the central path and the NT-search directions. Section 3 presents details concerning the parametric kernel function and barrier function. Section 4 presents the complexity results of small-and large-update algorithms for SDO, and some numerical results. Finally, some concluding remarks follow in Section 5.

These are some notes that have been used throughout this paper. The sets of real, nonnegative real, and positive real vectors with n components are denoted by $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$, respectively. E denoted an $n\times n$ identity matrix. The trace of a $n\times n$ matrix X is denoted by $tr(X) = \sum_{i=1}^n X_{ii}$, $tr(X^0Y^0) = n$, $\mathbf{S}^n$, $\mathbf{S}_+$, and $\mathbf{S}^n_{++}$ denote the cone of symmetric, symmetric positive semidefinite, and symmetric positive definite $n\times n$ matrices, respectively. For $m\in\mathbb{R}$, $\lceil m\rceil$ denotes the smallest integer greater than or equal to m.

2. PRELIMINARIES

2.1. Central path and classical NT search direction for SDO.

In this section, we define the central path and get the Nesterov-Todd search direction for SDO. We suppose (P) and (D) meet the interior point condition (IPC), i.e., there exists a $\left(X^0\succ 0, y^0, S^0\succ 0\right)$. Moreover, we may assume that $X^0 = S^0 = \mathbf{E}$, where $\mathbf{E}$ is the $n\times n$ identity matrix such that

$$A_i\bullet X^0 = b_i,\ 1\le i\le m,\ \sum_{i=1}^m y_i^0 A_i + S^0 = C,\ X_0\succ 0,\ S^0\succ 0.$$

The search for an optimum solution for problems (P) and (D) is similar to the resolution of the following system.

$$\begin{cases} A_i \bullet X = b_i, \ \ 1 \leq i \leq m, \ X \succeq 0, \\ \sum\limits_{i=1}^{m} y_i A_i + S = C, \ \ S \succeq 0, \\ XS = 0. \end{cases} \tag{2}$$

The key principle of primal-dual IPM is to replace the complementarity condition in (2) with the parameterized equation $XS = \mu\mathbf{E}$ $(\mu > 0)$. This provides the next system.

$$\begin{cases} A_i \bullet X = b_i, \quad 1 \leq i \leq m, \ \ X \succeq 0, \\ \sum\limits_{i=1}^{m} y_i A_i + S = C, \quad S \succeq 0, \\ XS = \mu\mathbf{E}. \end{cases} \tag{3}$$

According to the hypotheses, the system (3) has a unique solution $(X(\mu), y(\mu), S(\mu))$ for any $\mu > 0$.

The central path or central trajectory [9] is the set of μ-centers, with $\mu > 0$. If $\mu \longrightarrow 0$, the central path limit provides an optimum solution for (P) and (D) [2]. Therefore, we use Newton's approach to the system (3) to determine the search direction for SDO. We get

$$\begin{cases} A_i \bullet \Delta X = 0, \ 1 \leq i \leq m, \ X \succeq 0, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \ S \succeq 0, \\ X\Delta S + \Delta X S = \mu\mathbf{E} - XS. \end{cases} \tag{4}$$

It is clear that ΔS is symmetric due to the second equation in (4).

However it is important to note that ΔX is not necessarily symmetrical.

Many researchers have found various ways to render the third equation of the Newton system above (4)

symmetrical so that the new system that is made has only one symmetric solution.

Different choices of symmetrizing the third equation of (4) are proposed. In this work , we use the NT-symmetrization scheme [10]. Let's

$$P := X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = S^{-\frac{1}{2}}\left(S^{\frac{1}{2}}XS^{\frac{1}{2}})\right)^{\frac{1}{2}}S^{-\frac{1}{2}},$$

Moreover, also define $D = P^{\frac{1}{2}}$, the matrix D can rescale X and S to the same matrix V, defined by

$$V = \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} (D^{-1} XSD)^{\frac{1}{2}}. \tag{5}$$

Note that the matrices D and V are symmetric and positive definite.

From (5), and after a few simple reductions, the NT search direction $(D_X, \Delta y, D_S)$ matches the following system:

$$\begin{cases} \bar{A}_i \bullet D_X = 0, \;\; 1 \leq i \leq m, \\ \sum\limits_{i=1}^{m} \Delta y_i \bar{A}_i + D_S = 0, \\ D_X + D_S = V^{-1} - V. \end{cases} \tag{6}$$

with

$$\begin{aligned} &\bar{A}_i = \tfrac{1}{\sqrt{\mu}} D A_i D, \quad 1 \leq i \leq m, \\ &D_X = \tfrac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \quad D_S = \tfrac{1}{\sqrt{\mu}} D \Delta S D \end{aligned} \tag{7}$$

The unique solution $(D_X, \Delta y, D_S)$ of the system (6) is called NT search direction. The first two equations in the system (6) imply that D_X and D_S are orthogonal:

$$\operatorname{tr}(D_X D_S) = \operatorname{tr}(D_S D_X) = 0.$$

Let us review some fundamental concepts and result-related matrix functions $[11, 12]$ that will be used to define the algorithm.

Definition 2.1. Let $V \in \mathbf{S}^n_{++}$ and

$$V = Q^T diag(\lambda_1(V), \lambda_2(V), \cdots, \lambda_n(V)) Q,$$

where Q is any orthonormal matrix that diagonalizes V, and let $\psi(t)$ be defined as in (1). The matrix valued-function $\psi(V) : \mathbf{S}^n_{++} \longrightarrow \mathbf{S}^n$ is defined by

$$\psi(V) = Q^T diag\left(\psi(\lambda_1(V)), \psi\left(\lambda_2(V)\right), \ldots, \psi\left(\lambda_n(V)\right)\right) Q. \tag{8}$$

Note that $\psi(V)$ depends only on the restriction of $\psi(t)$ to the set of eigenvalues of V. Assume that $\psi(t)$ is twice differentiable, for $t > 0$, the derivatives $\psi'(t)$ and $\psi''(t)$ are well-defined. As a result, by replacing $\psi(\lambda_i(V))$ in (11) with $\psi'(\lambda_i(V))$ and $\psi''(\lambda_i(V))$, we obtain that the matrix functions $\psi'(V)$ and $\psi''(V)$ are also defined. Like the case LO, the real-valued matrix function $\Psi(V)$ is defined as follows.

Definition 2.2. We define $\Psi(V) : \mathbf{S}^n_{++} \to \mathbb{R}_+$ by:

$$\Psi(V) = \operatorname{tr}(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)), \tag{9}$$

where $\psi(V)$ is given by (8).

Referring to $[8,13]$, the right side of the third equation of (6), $V^{-1} - V$, is replaced by $-\psi'(V)$. Therefore, this system might be rebuilt as follows:

$$\begin{cases} \bar{A}_i \bullet D_X = 0, \ \ 1 \leq i \leq m, \\ \sum\limits_{i=1}^{m} \Delta y_i \bar{A}_i + D_S = 0, \\ D_X + D_S = -\psi'(V). \end{cases} \tag{10}$$

By taking a default step size α along the search direction, we construct a new triple $(X_+, y_+, \ S_+)$ according to

$$X_+ = X + \alpha \Delta X, \ y_+ = y + \alpha \Delta y, \ S_+ = S + \alpha \Delta S. \tag{11}$$

It is easy to check that.

$$\Psi(V) = 0 \Leftrightarrow V = \mathbf{E} \Leftrightarrow D_X = D_S = 0_{n\times n} \Leftrightarrow X = X(\mu), S = S(\mu).$$

The algorithm corresponding to the primal-dual IPM for the SDO is based on our kernel function summarized in Algorithm 1

Algorithm 1. Generic Interior Point Algorithm for SDO

Input

a threshold parameter $\tau \geq 1$;

an accuracy parameter $\varepsilon > 0$;

a fixed barrier update parameter θ, $0 < \theta < 1$;

$X^0 \succ 0$, $S^0 \succ 0$ and $\mu^0 = 1$ such that $\Psi\left(X^0, S^0, \mu^0\right) \leq \tau$.

begin

$X := X^0; S = S^0; \mu = \mu^0$

while $n\mu \geq \varepsilon$ do

begin

$\mu := (1-\theta)\mu$

while $\Psi(X,S,\mu) \geq \tau$ do

begin

Solve system (10) and use (7) to obtain $(\Delta X,\ \Delta y,\ \Delta S)$

Determine a suitable step size α

Update $(X,y,S) := (X,y,S) + \alpha(\Delta X, \Delta y, \Delta S)$.

end

end

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3. Properties of the Kernel Function and the Barrier Function

First, we study the essential characteristics of the kernel function. $\psi(t)$. The precise details are given in [5].

3.1. Kernel function properties. We begin with some characteristics. For ψ, the first three derivatives are as follows:

$$\psi'(t) = 2(p+2)t - (p+3) - \frac{p}{t^{p+1}} - \frac{1}{t}, \tag{12}$$

$$\psi''(t) = 2(p+2) + \frac{1}{t^2} + \frac{p(p+1)}{t^{p+2}}, \text{ for } t > 0,\ p \geq 2, \tag{13}$$

$$\psi'''(t) = -\left(\frac{p(p+1)(p+2)}{t^{p+3}} + \frac{2}{t^3}\right). \tag{14}$$

From (13), we have

$$\psi''(t) > 2(p+2), \quad p \geq 2, \quad t > 0. \tag{15}$$

It follows that $\psi(1) = \psi'(1) = 0$. Moreover, it is easy to confirm that

$$\lim_{t \longrightarrow 0^+} \psi(t) = \lim_{t \longrightarrow \infty} \psi(t) = \infty.$$

Due to the conditions $\psi(1) = \psi'(1) = 0$, it is possible to describe $\psi(t)$ entirely by its second derivative.

$$\psi(t) = \int_1^t \int_1^x \psi''(y)\, dydx. \tag{16}$$

The following lemma helps prove that (1) is an effective kernel function.

Lemma 3.1. *Let $\psi(t)$ be as defined in (1). Then*

a) $\psi''(t) > 1$,

b) $\psi'''(t) < 0$,

c) $t\psi''(t) + \psi'(t) > 0$,

d) $t\psi''(t) - \psi'(t) > 0$.

Proof. Inequalities $a)$ and $b)$ immediately follows from (13) and (14), respectively. Next, we prove that (c) holds. Using (13) and $p \geq 2$

$$t\psi''(t) + \psi'(t) = (4p+8)t + \frac{p^2}{t^{p+1}} - (p+3).$$

Let

$$h(t) = (4p+8)t + \frac{p^2}{t^{p+1}} - (p+3).$$

Then

$$\begin{aligned} h'(t) &= \frac{-p^2(p+1)}{t^{p+2}} + 4p + 8, \\ h''(t) &= \frac{p^2(p+1)(p+2)}{t^{p+3}} > 0, \text{ for all } t > 0. \end{aligned}$$

Let $h'(t) = 0$, we get $t_p = \left(\frac{p^2(p+1)}{4p+8}\right)^{\frac{1}{p+2}}$. Because $h(t)$ is strictly convex and has a global minimum, $h(t_p) > 0$. We have the result. Furthermore, for proving (d) we have

$$t\psi''(t) - \psi'(t) = p + 3 + \frac{p}{t^{p+1}}(p+2) + \frac{2}{t} > 0, \quad p \geq 2.$$

This completes the proof. □

Lemma 3.2 (*Proposition* 3 in [14])**.** *For any $V_1, V_2 \succ 0$,*

$$\Psi\left(\left[V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}\right]^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi(V_1) + \Psi(V_2)\right).$$

As an observation, we offer some results about the new kernel function.

Lemma 3.3. *For $\psi(t)$, the following results hold:*

$$(p+2)(t-1)^2 \leq \psi(t) \leq \frac{1}{4(p+2)}\psi'(t)^2, \quad t>0, \tag{17}$$

$$\frac{\psi'(t)}{2}(t-1) \leq \psi(t) \leq \frac{\psi''(1)}{2}(t-1)^2, \quad t \geq 1, \tag{18}$$

Proof. For, (17), it obtained by using (15) and (16) for all $t>0$, then we have

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^x \psi''(y)\,dydx \geq 2(p+2)\int_1^t \int_1^x dydx \\ &= 2(p+2)\int_1^t (x-1)\,dx = (p+2)(t-1)^2,\end{aligned}$$

this demonstrates the first inequality. The second inequality may be calculated as follows

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^x \psi''(y)\,dydx \leq \frac{1}{2(p+2)}\int_1^t \int_1^x \psi''(x)\,\psi''(y)\,dydx \\ &= \frac{1}{2(p+2)}\int_1^t \psi''(x)\,\psi'(x)\,dx = \frac{1}{4(p+2)}\left[\psi'(t)\right]^2.\end{aligned}$$

For (18) if $f(t) = 2\psi(t) - (t-1)\,\psi'(t)$, then $f'(t) = \psi'(t) - \psi''(t)(t-1)$, $f''(t) = -(t-1)\,\psi'''(t)$ and $f(1) = f'(1) = 0$. Since $\psi'''(t) < 0$, we deduce that $f''(t) \geq 0$, which implies that f' is increasing. Thus, $f'(t) \geq 0$ for $t \geq 1$. Similarly, $f(t) \geq 0$. This proves left inequality.

To prove right inequality, by using Taylor's development and the fact $\psi(1) = \psi'(1) = 0$, $\psi'''(t) < 0$, $\psi''(1) = p^2+3p+5$, we have for some ξ, such that $1 \leq \xi \leq t$.

$$\begin{aligned}\psi(t) &= \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi^{(3)}(\xi)(\xi-1)^3 \\ &\leq \frac{(p^2+3p+5)}{2}(t-1)^2.\end{aligned}$$

□

Remark 1. Let $g(t) = (p+3)t - \frac{1}{t^p} + \log(t)$, $t>0$. Then $\psi(t) = -g(t) + (p+2)t^2$ since $g'(t) = (p+3) + \frac{p}{t^{p+1}} + \frac{1}{t} > 0$, $g(t)$ is monotically increazing with respect to $t \geq 1$.

Let $\rho : [0,+\infty) \longrightarrow [1,+\infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$ and $\rho : [0,+\infty) \longrightarrow (0,1]$ the inverse function of $-\frac{\psi'(t)}{2}$ restricted to the interval $(0,1]$. This leads us to the next lemma.

Lemma 3.4. *We have*

$$\sqrt{\frac{u}{p+2}+1} \leq \rho \leq \sqrt{\frac{u}{p+2}}+1, \ u \geq 0. \tag{19}$$

$$\rho(z) \geq \left(\frac{p}{2z+p}\right)^{\frac{1}{p+1}}, \qquad z \geq 0. \tag{20}$$

Proof. For (19), let $u = \psi(t)$ $t \geq 1$. Then $\rho(u) = t$, using (17) we have $u = \psi(t) \geq (p+2)(t-1)^2$, so

$$t = \rho(u) \leq \sqrt{\frac{u}{p+2}}+1.$$

By the definition of $\psi(t)$ we have $u = \psi(t) = -g(t) + (p+2)t^2$. Using Remark 2 and $g(1) = p+2$, we have $g(t) \geq p+2$, $t \geq 1$. Hence, $t^2 = \frac{u+g(t)}{p+2} \geq \frac{u+p+2}{p+2} = 1 + \frac{u}{p+2}$.

This implies that

$$t = \rho(u) \geq \sqrt{1+\frac{u}{p+2}}.$$

For (20). Let $z = -\frac{\psi'(t)}{2}$ for $0 < t \leq 1$. Due to the definition of ρ, $\rho(z) = t$, $z \geq 0$. and $2z = -\psi'(t)$, we have $\frac{p}{t^{p+1}} = 2z + 2(p+2)t - \frac{1}{t} - p - 3$ because $g : t \longrightarrow 2(p+2)t - \frac{1}{t} - p - 3$ is monotone increasing with respect to $t \in (0,1]$ $g'(t) = 2p + 4 + \frac{1}{t^2} > 0$, $g(1) = p$ and hence, $\frac{p}{t^{p+1}} \leq 2z + p$. That is

$$\rho(z) = t \geq \left(\frac{p}{p+2z}\right)^{\frac{1}{p+1}}.$$

This completes the proof. □

Lemma 3.5. *If* $\beta \geq 1$. *Then*

$$\psi(\beta t) \leq \psi(t) + (p+2)\left(\beta^2 - 1\right)t^2.$$

Proof. Using Remark 2 we have $g(\beta t) - g(t) \geq 0$ for $\beta \geq 1$.

Hence

$$\begin{aligned}
\psi(\beta t) &= (p+2)\beta^2 t^2 - g(\beta t) + (p+2)t^2 + g(t) - (p+2)t^2 - g(t) \\
&= (p+2)\beta^2 t^2 - g(\beta t) + \psi(t) + g(t) - (p+2)t^2 \\
&= \psi(t) + (p+2)\left(\beta^2 - 1\right)t^2 - (g(\beta t) - g(t)) \\
&\leq \psi(t) + (p+2)\left(\beta^2 - 1\right)t^2.
\end{aligned}$$

This completes the proof. □

4. Algorithm Analysis for SDO

In this, we examine the complexity of the SDO interior-point algorithm for both small and large updates. The algorithm is analyzed using the norm-based proximity measure.

$$\delta(V) := \frac{1}{2}\left\|\Psi'(V)\right\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}\left[\psi'(\lambda_i(V))\right]^2} = \frac{1}{2}\left\|D_X + D_S\right\|. \tag{21}$$

The following lemma explains the relationship between two proximity measures.

Lemma 4.1. *Let $\delta(V)$ be defined as in (21). Then we have*

$$\delta(V) \geq \sqrt{(p+2)\Psi(V)},\ V \in \mathbf{S}^n_{++},\ p \geq 2.$$

Proof. Using the second inequality of (17) we have

$$\begin{aligned}\Psi(V) &= \operatorname{tr}(\psi(V)) = \sum_{i=1}^{n}\psi(\lambda_i(V)) \leq \frac{1}{4(p+2)}\sum_{i=1}^{n}\left[\psi'(\lambda_i(V))\right]^2 \\ &= \frac{1}{4(p+2)}\sum_{i=1}^{n}\|\nabla\Psi(V)\|^2 = \frac{1}{p+2}\delta^2(V).\end{aligned}$$

Hence, we have

$$\delta(V) \geq \sqrt{(p+2)\Psi(V)}.$$

This completes the proof. □

Remark 2. We always make the following suppositions $\tau \geq 1$. Using Lemma 6 and the assumption that $\Psi(v) \geq \tau$, we have

$$\delta(V) \geq \sqrt{p+2}.$$

The following theorem is an extension of (Theorem 3.2 in [4],) to positive definite matrices.

Theorem 4.2 (*Theorem* 3 in [15])**.** *Let $\rho : [0,\infty[\to [1,\infty[$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then we have for any positive definite matrix V and any $\beta \geq 1$:*

$$\Psi(\beta V) \leq n\psi\left(\beta\rho\left(\frac{\Psi(V)}{n}\right)\right).$$

Lemma 4.3. *Let* $0 \leq \theta < 1$ *and* $V_+ := \frac{V}{\sqrt{1-\theta}}$. *If* $\Psi(V) \leq \tau$, *then for* $p \geq 2$, *we have*

$$\Psi(V_+) \leq \frac{p^2+3p+5}{2(1-\theta)}\left[\theta\sqrt{n}+\sqrt{\frac{\tau}{\alpha+2}}\right]^2. \tag{22}$$

$$\Psi(V_+) \leq \frac{\tau+n(p+2)\theta+2\theta\sqrt{n(p+2)\tau}}{1-\theta}. \tag{23}$$

Proof. To prove (22) of the lemma, since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\rho\left(\frac{\Psi(V)}{n}\right) \geq 1$, we have $\frac{\rho\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Theorem 1 with $\beta = \frac{1}{\sqrt{1-\theta}}$, the inequality (18) in Lemma 3 and $\Psi(V) \leq \tau$, we have

$$\begin{aligned}
\Psi(V_+) &\leq n\psi\left(\frac{\rho\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}\right) \leq \frac{n\left(p^2+3p+5\right)}{2}\left[\frac{\rho\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}-1\right]^2 \\
&\leq \frac{n\left(p^2+3p+5\right)}{2(1-\theta)}\left[\sqrt{\frac{\Psi(V)}{n(p+2)}}+1-\sqrt{1-\theta}\right]^2 \\
&= \frac{n\left(p^2+3p+5\right)}{2(1-\theta)}\left[\sqrt{\frac{\Psi(V)}{n(p+2)}}+\frac{\theta}{1+\sqrt{1-\theta}}\right]^2 \\
&\leq \frac{p^2+3p+5}{2(1-\theta)}\left[\sqrt{\frac{\tau}{p+2}}+\sqrt{n}\theta\right]^2.
\end{aligned}$$

Where the last inequality holds from $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, $0 \leq \theta < 1$. For (23), using Theorem 1 with $\beta = \frac{1}{\sqrt{1-\theta}}$, $\Psi(V) \leq \tau$ inequality (20) and Lemma 5 we obtain the other upper bound of $\Psi(V)$ as follows:

$$\begin{aligned}
\Psi(V_+) &\leq n\psi\left(\frac{\rho\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\left(\psi\left(\rho\left(\frac{\Psi(V)}{n}\right)\right)+\frac{\theta(p+2)}{1-\theta}\rho^2\left(\frac{\Psi(V)}{n}\right)\right) \\
&= \Psi(V)+\frac{n\theta}{1-\theta}(p+2)\rho^2\left(\frac{\Psi(V)}{n}\right) \leq \Psi(V)+\frac{n\theta}{1-\theta}(p+2)\left[\sqrt{\frac{\frac{\Psi(V)}{n}}{(p+2)}}+1\right]^2 \\
&\leq \tau+\frac{n\theta}{1-\theta}(p+2)\left[\sqrt{\frac{\tau}{n(p+2)}}+1\right]^2 = \frac{\tau+n(p+2)\theta+2\theta\sqrt{n(p+2)\tau}}{1-\theta}.
\end{aligned}$$

This completes the proof. $\square$

Denote

$$\begin{aligned}
\bar{\Psi}_0 &= \frac{\tau+n(p+2)\theta+2\theta\sqrt{n(p+2)\tau}}{1-\theta}, \\
\tilde{\Psi}_0 &= \frac{p^2+3p+5}{2(1-\theta)}\left[\sqrt{\frac{\tau}{p+2}}+\sqrt{n}\theta\right]^2.
\end{aligned} \tag{24}$$

We will use $\bar{\Psi}_0$ and $\tilde{\Psi}_0$ for the upper bounds of $\Psi(V)$ for large-update and small-update methods respectively during the process of the algorithm.

Remark 3. For the large-update method, by taking $\tau = \mathscr{O}(n)$, $\theta = \Theta(1)$, we have $\bar{\Psi}_0 = \mathscr{O}(pn)$, for small-update methods with
$\tau = \mathscr{O}(1)$, $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, we have $\tilde{\Psi}_0 = \mathscr{O}(1)$.

4.1. Step size determination. We can write from (11) and (7)

$$X_+ := X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V + \alpha D_X)D,$$

and

$$S_+ := S + \alpha\Delta S = S + \alpha\sqrt{\mu}D^{-1}D_SD^{-1} = \sqrt{\mu}D^{-1}(V + \alpha D_S)D^{-1}.$$

Thus we have

$$V_+^2 = (V + \alpha D_X)(V + \alpha D_S)$$

Therefore, V_+^2 is similar to the matrix

$$\left[(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}\right].$$

Consequently, the eigenvalues of the matrix V_+ are the same as those of

$$\left[(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

Since the proximity after one step is defined by $\Psi(V_+)$, and then we have

$$\Psi(V_+) = \Psi\left(\left[(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right).$$

By Theorem 4, we obtain

$$\Psi(V_+) \le \frac{1}{2}\left[\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right].$$

Define for $\alpha > 0$,

$$f(\alpha) := \Psi(V_+) - \Psi(V).$$

According to Lemma 2 and the definition of $f(\alpha)$, we obtain that $f(\alpha) \le f_1(\alpha)$ where

$$f_1(\alpha) := \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right) - \Psi(V).$$

Obviously

$$f(0) = f_1(0) = 0.$$

We obtain by taking the derivative with respect to α

$$f_1'(\alpha) = \frac{1}{2}\mathrm{tr}\left(\psi'(V+\alpha D_X)D_X + \psi'(V+\alpha D_S)D_S\right),$$

and

$$\begin{aligned} f_1''(\alpha) &= \frac{1}{2}\frac{d^2}{d\alpha^2}\mathrm{tr}\left(\psi(V+\alpha D_X) + \psi(V+\alpha D_S)\right) \\ &= \frac{1}{2}\mathrm{tr}\left(\psi''(V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_S)D_S^2\right). \end{aligned}$$

It is evident that $f_1(\alpha) > 0$ unless $D_X = D_S = 0$.

Hence, using (21) and the third equation of (10), we obtain

$$\begin{aligned} f_1'(0) &= \frac{1}{2}\mathrm{tr}[\psi(V)'D_X + \psi(V)'D_S] = \frac{1}{2}\mathrm{tr}\left(\psi'(V)(D_X+D_S)\right) \\ &= \frac{1}{2}\mathrm{tr}\left[\psi(V)'\left(-\psi'(V)\right)\right] = \frac{1}{2}\mathrm{tr}\left(-\psi'(V)^2\right) = -2\delta^2(V). \end{aligned} \tag{25}$$

The notations used in the sequel are as follows: $\delta := \delta(V)$ and $\Psi := \Psi(V)$.

Lemma 4.4 (*Lemma* 8 in [15])**.** *Let δ be defined as in* (21). *Then we have*

$$f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_n(V) - 2\alpha\delta),$$

where $\lambda_n(V)$ is the smallest eigenvalue of V.

Using Lemma 8 and (25), we have the following lemma.

Lemma 4.5 (*Lemma* 4.2 in [4])**.** *If the step size α satisfies*

$$\psi'(\lambda_n(V)) - \psi'(\lambda_n(V) - 2\alpha\delta) \leq 2\delta, \tag{26}$$

then

$$f'(\alpha) \leq 0.$$

Lemma 4.6 (*Lemma* 4.3 in [4])**.** *Let $\rho : [0,\infty) \to (0,1]$ denote the inverse function of the restriction of -$\frac{1}{2}\psi'(t)$ on the interval $(0,1]$, then the largest possible value of the step size of α satisfying* (26) *is given by*

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta).$$

Lemma 4.7 (*Lemma* 4.4 in [4])**.** *Let ρ and $\bar{\alpha}$ be the same as defined in Lemma* 10*. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

In the sequel, we use the notation

$$\bar{\alpha} = \frac{1}{\psi''(\rho(2\delta))} \tag{27}$$

Lemma 4.8. *Let ρ and $\bar{\alpha}$ be as defined in Lemma* 11*. If $\Psi(v) \geq \tau \geq 1$, then we have*

$$\bar{\alpha} \geq \frac{\delta^{-\frac{p+2}{p+1}}}{25p^2 + 27p + 29}.$$

Proof. Applying Lemma 11, the definition of $\psi''(t)$, and (20) we have $\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}$, and for $\rho(2\delta) = t \in (0,1]$ we have

$$\frac{1}{\psi''(\rho(2\delta))} = \frac{1}{2(p+2) + \frac{p(p+1)}{(\rho(2\delta))^{p+2}} + \frac{1}{\rho(2\delta)^2}},$$

for $z = 2\delta$, this implies that

$$\psi''(\rho(2\delta)) \leq 2(p+2) + p(p+1)\left(\frac{4\delta + p}{p}\right)^{\frac{p+2}{p+1}} + \left(\frac{4\delta + p}{p}\right)^{\frac{2}{p+1}},$$

we have

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{2(p+2) + p(p+1)\left(\frac{4\delta+p}{p}\right)^{\frac{p+2}{p+1}} + \left(\frac{4\delta+p}{p}\right)^{\frac{2}{p+1}}}$$

Since $(4\delta+1)^{\frac{2}{p+1}} \leq (4\delta+1)^{\frac{p+2}{p+1}}$ for $p \in [2,\infty[$, it follows that

$$\bar{\alpha} \geq \frac{1}{2(p+2) + (p^2+p+1)\left(\frac{4\delta+p}{p}\right)^{\frac{p+2}{p+1}}}$$

Using Remark 3

$$\begin{aligned}
\bar{\alpha} &\geq \frac{1}{2\sqrt{p+2}\delta+\left(p^2+p+1\right)\left(\frac{4\delta+p}{p}\right)^{\frac{p+2}{p+1}}} \\
&\geq \frac{1}{\left[2\left(p+2\right)+\left(p^2+p+1\right)5^{\frac{p+2}{p+1}}\right]\delta^{\frac{p+2}{p+1}}} \\
&\geq \frac{\delta^{-\frac{p+2}{p+1}}}{25p^2+27p+29}.
\end{aligned}$$

This completes the proof. □

For using $\tilde{\alpha}$ as the default step size in the algorithm, define the $\tilde{\alpha}$ as follows

$$\tilde{\alpha} = \frac{\delta^{-\frac{p+2}{p+1}}}{25p^2+27p+29}. \tag{28}$$

4.2. Decrease the value of $\Psi(V)$ during an inner iteration. Now we demonstrate that, with our default step size α, our proximity function is diminishing.

Lemma 4.9 (*Lemma* 4.5 in [4])**.** *If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Lemma 4.10. *Since the default size $\tilde{\alpha}$ satisfies $\tilde{\alpha} \leq \bar{\alpha}$, by Lemma* 13*. We have the following upper bound for $f(\tilde{\alpha})$:*

$$f(\tilde{\alpha}) \leq -\frac{(p+1)^{\frac{p}{2(p+1)}-1}}{25}\Psi^{\frac{p}{2(p+1)}} \tag{29}$$

Proof. Using Lemma 13 with $\alpha = \tilde{\alpha}$ (28), we have

$$\begin{aligned}
f(\tilde{\alpha}) &\leq -\tilde{\alpha}\delta^2 = -\frac{\delta^{2-\frac{p+2}{(p+1)}}}{25p^2+27p+29} \\
&= -\frac{\delta^{\frac{p}{p+1}}}{25p^2+27p+29} \\
&\leq -\frac{(p+2)^{\frac{p}{2(p+1)}}\Psi^{\frac{p}{2(p+1)}}}{25\left(p^2+\frac{27}{25}p+\frac{29}{25}\right)} \leq -\frac{(p+2)^{\frac{p}{2(p+1)}}\Psi^{\frac{p}{2(p+1)}}}{25\left(p+1\right)^2} \\
&\leq -\frac{(p+1)^{\frac{p}{2(p+1)}-1}\Psi^{\frac{p}{2(p+1)}}}{25}.
\end{aligned}$$

This proves the theorem. □

Proposition 4.11 (Proposition 1.3.2 in [14])**.** *Suppose that a sequence* $\left\{t^k > 0,\ k = 0,1,2,...,K\right\}$ *is satisfying the following inequality:*

$$t_{k+1} \le t_k - \eta t_k^{1-\gamma}, \quad k = 0,1,2,...,K-1,$$

where $\eta > 0$ *and* $\gamma \in (0,1]$. *Then* $K \le \left\lceil \dfrac{t_0^{\gamma}}{\eta\gamma} \right\rceil$.

The value of Ψ after-update is expressed as Ψ_0, and the successive values in same outer iteration are represented as Ψ_l, $l = 0,1,2,3,...,\ K$, where K denotes the total number of inner iterations per an outer iteration. Then we have $\Psi_0 \le \tilde{\Psi}_0$ and $\Psi_0 \le \bar{\Psi}_0$, where $\tilde{\Psi}_0$ and $\bar{\Psi}_0$ are defined in(24). Then we have $\Psi_{K-1} > \tau$ and $0 \le \Psi_K \le \tau$.

(29) shows the diminution of every inner iteration. In [4] we may obtain the proper values of η and $\gamma \in (0,1]$.

$$\eta = \frac{1}{25}(p+1)^{\frac{p}{2(p+1)}-1} \text{ and } \gamma = \frac{p+2}{2(p+1)}.$$

Lemma 4.12. *Let* L_1 *and* L_2 *be the total numbers of inner iterations in the outer iteration for small- and large-update methods, respectively. Then for* $p \ge 2$, *we have*

$$L_1 \le \left\lceil 50(p+1)^{\frac{p+2}{2(p+1)}} \tilde{\Psi}_0^{\frac{p+2}{2(p+1)}} \right\rceil, \tag{30}$$

$$L_2 \le \left\lceil 50(p+1)^{\frac{p+2}{2(p+1)}} \bar{\Psi}_0^{\frac{p+2}{2(p+1)}} \right\rceil, \tag{31}$$

Proof. For (30), using Proposition 1 and Lemma 15 with $\eta = \frac{1}{25}(p+1)^{\frac{p}{2(p+1)}-1}$ and $\gamma = \frac{p+2}{2(p+1)}$ we get:

$$L_1 \le \left\lceil \frac{\Psi_0^{\gamma}}{\eta\gamma} \right\rceil = \left\lceil \frac{50(p+1)^{\frac{3p+4}{2(p+1)}} \tilde{\Psi}_0^{\frac{p+2}{2(p+1)}}}{p+2} \right\rceil \le \left\lceil 50(p+1)^{\frac{p+2}{2(p+1)}} \tilde{\Psi}_0^{\frac{p+2}{2(p+1)}} \right\rceil$$

For (31), in a similar fashion, we have

$$L_2 \le \left\lceil 50(p+1)^{\frac{p+2}{2(p+1)}} \bar{\Psi}_0^{\frac{p+2}{2(p+1)}} \right\rceil$$

This completes the proof. $\square$

The number of barrier parameter updates is given by (cf. 11, Lemma II.7 page 116)

$$\left\lceil \frac{1}{\theta} \log \frac{n}{\varepsilon} \right\rceil.$$

By multiplying the number of outer iterations by the number of inner iterations, the total number of iterations for small-and large-update methods is bounded by

$$\left\lceil 50\,(p+1)^{\frac{p+2}{2(p+1)}}\,\bar{\Psi}_0^{\frac{p+2}{2(p+1)}}\,\frac{1}{\theta}\log\frac{n}{\varepsilon}\right\rceil,\ \text{and}\ \left\lceil 50\,(p+1)^{\frac{p+2}{2(p+1)}}\,\tilde{\Psi}_0^{\frac{p+2}{2(p+1)}}\,\frac{1}{\theta}\log\frac{n}{\varepsilon}\right\rceil$$

For large-update methods with one takes for θ a constant (independent on n), namely $\theta = \Theta(1)$, and $\tau = \mathscr{O}(n)$, The iteration bound then becomes

$$\mathscr{O}\left((p+1)^{\frac{p+2}{2(p+1)}}\,n^{\frac{p+2}{2(p+1)}}\,\log\frac{n}{\varepsilon}\right).$$

For small-update methods, we have $\tau = \mathscr{O}(1)$ and $\theta = \frac{1}{\sqrt{n}}$. A better bound is obtained by using the upper bound $\tilde{\Psi}_0$ in (24). Note now $\tilde{\Psi}_0 = \mathscr{O}(p)$, and the iteration bound becomes

$$\mathscr{O}\left((p+1)^{\frac{p+2}{2(p+1)}}\,\sqrt{n}\log\frac{n}{\varepsilon}\right)\ \text{iterations complexity.}$$

Remark 4. $p = 2$ is also an appropriate choice. The iteration bound is then $\mathscr{O}\left(n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right)$ and $\mathscr{O}\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$ iterations complexity for large and small-update methods, respectively.

4.3. Numerical tests. The following examples are taken from [16] and implemented in the C++ language. We have taken $\varepsilon = 10^{-6}$, $\tau = n$, and $\theta \in (0,1)$. In the table of results, (m,n) represents the size of the problem, and **Itr** is the number of iterations necessary to obtain an optimal solution. In our proposed kernel function, we have taken $p = 2$ and we compare our numerical results with the alternatives proposed in [16].

We consider the LSO problem:

$$(P)\qquad \min\left\{C \bullet X,\ A_i \bullet X = b_i,\ 1 \le i \le m,\ X \succeq 0\right\},$$

and its dual problem

$$(D)\qquad \max\left\{b^T y:\ \sum_{i=1}^{m} y_i A_i + S = C,\ S \succeq 0\right\},$$

Example 4.13. : $C(i,j) = -1\ \forall i,j = 1,2$, $A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $A_2 = \mathbf{E}$, and $b = (1,1)^T$. The initial solution (X_0, y_0, S_0), such that $X_0 = diag\left(\frac{1}{2}, -\frac{1}{2}\right)$, $y_0 = (0,-3)^T$ and $S_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

	Numerical results				
	Our Algorithm	Algorithm [16]			
	Itr	**Itr** (alt 1)	**Itr** (alt 2)	**Itr** (alt 3)	**Itr** (alt 4)
Small update Alg	4	5	5	5	5
Large update Alg	2	-	-	-	-

Example 4.14. $C = diag(5,8,5,5)$, $A_4 = \mathbf{E}$, $b = (1,1,1,2)^T$ *and*

$$A_k(i,j) = \begin{cases} 1 & \text{if } i = j = k \text{ or } i = j = k+1 \\ -1 & \text{if } i = k,\ j = k+1 \text{ or } i = k+1,\ j = k,\ k = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

The initial solution (X_0, y_0, S_0), such that $X_0 = \frac{1}{2}\mathbf{E}$, $y_0 = (1.5, 1.5, 1.5, 1.5)^T$ and $S_0 =$
$$\begin{pmatrix} 2 & 1,5 & 0 & 0 \\ 1,5 & 3,5 & 1,5 & 0 \\ 0 & 1,5 & 3,5 & 1,5 \\ 0 & 0 & 1,5 & 2 \end{pmatrix}$$

	Numerical results				
	Our Algorithm	Algorithm [16]			
	Itr	**Itr** (alt 1)	**Itr** (alt 2)	**Itr** (alt 3)	**Itr** (alt 4)
Small update Alg	9	14	7	7	7
Large update Alg	6	-	-	-	-

Example 4.15. $C = diag(-4,-2,-2,0,0,0)$, $A_1 = diag(1,-1,1,1,0,0)$,

$A_2 = diag(1,1,1,0,1,0)$, $A_3 = diag(2,2,1,0,0,1)$ and $b = (6,2,4)^T$

The initial solution (X_0, y_0, S_0), such that

$X_0 = diag(1.467, 0.087, 0.36, 4.26, 0.086, 0.532)$, $y_0 = (-1,-1,-2)^T$ and $S_0 = diag(2,2,2,1,1,2)$,

	Numerical results				
	Our Algorithm	Algorithm [16]			
	Itr	**Itr** (alt 1)	**Itr** (alt 2)	**Itr** (alt 3)	**Itr** (alt 4)
Small update Alg	11	18	8	8	6
Large update Alg	5	-	-	-	-

Example 4.16. (m,n) : variable size, $C=\mathbf{E}$, the matrices $A_k, k=1,\dots,m$, are defined as follows

$$A_k = \begin{cases} 1 & \text{if} \quad i=j=k \\ 1 & \text{if} \quad i=j \text{ and } i=m+k, \\ 0 & \text{otherwise,} \end{cases}$$

and $b(i)=2, i=1,\dots,m$.

The initial solution (X^0,y^0,S^0), such that $X^0(x^0_{ij}) = \begin{cases} 1.5 & \text{if} \quad i\leq j \\ 0.5 & \text{if} \quad i>j \end{cases}$,

$y^0(i)=-2, i=1,\dots,m$,and $S^0=\mathbf{E}$.

		Numerical results				
		Our Algorithm	Algorithm [16]			
	size (m,n)	**Itr**	**Itr** (alt 1)	**Itr** (alt 2)	**Itr** (alt 3)	**Itr** (alt 4)
Small -update Alg	(10, 20)	4	4	4	8	4
	(20, 40)	4	4	4	8	4
	(50, 100)	3	4	4	9	4
	(100, 200)	3	5	5	9	5
	(150, 300)	3	6	6	9	5
Large -update Alg	(10, 20)	3	-	-	-	-
	(20, 40)	3	-	-	-	-
	(50, 100)	2	-	-	-	-
	(100, 200)	2	-	-	-	-
	(150, 300)	2	-	-	-	-

5. Conclusion

In this work, we propose a novel kernel function with a double barrier term and use it in conjunction with a primal-dual path-following interior point technique to solve semidefinite optimization problems. For large and small-update algorithms, the iteration bounds are $\mathscr{O}\left((pn)^{\frac{p+1}{2p}}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$ and $\mathscr{O}\left(p^{\frac{p+1}{2p}}\sqrt{n}\log\frac{\operatorname{tr}\left(X^0S^0\right)}{\varepsilon}\right)$, respectively. We established that the iteration bound of a large-update interior-point method is $\mathscr{O}\left(n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right)$, and we see that complexity has been reduced by a factor of $n^{\frac{1}{3}}$. For small-update methods, we have obtained $\mathscr{O}\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$ iteration bound which matches the currently best-known iteration bound for small-update methods. Future study might concentrate on the extension to symmetric cone optimization. Eventually, several strategies are used for numerical tests, indicating that the kernel function used in the algorithm is effective.

Conflict of Interests

The authors declare that there is no conflict of interests.

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