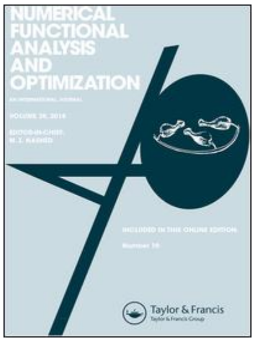

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On Numerical Radius Inequalities for Operator Matrices

Hanane Guelfen[a] and Fuad Kittaneh[b]

[a]Department of Mathematics, The University of Batna 2, Algeria; [b]Department of Mathematics, The University of Jordan, Amman, Jordan

ABSTRACT

The aim of this article is to prove several new numerical radius inequalities for $n \times n$ operator matrices on a Hilbert space. Let $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$ be complex Hilbert spaces, and let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Among other inequalities, it shown that

$$\omega(T) \leq \frac{1}{2}\sum_{i=1}^{n}\left(w(T_{ii}) + \sqrt{\omega^2(T_{ii}) + \sum_{\substack{j=1\\ j\neq i}}^{n} ||T_{ij}||^2}\right),$$

where $\omega(\cdot)$ and $||\cdot||$ denote the numerical radius and the usual operator norm, respectively.

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1. Introduction

Let $\mathbb{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bonded linear operators on $\mathbb{H}$, and for $T \in \mathbb{B}(\mathbb{H})$, let $r(T), \omega(T)$, and $||T||$ denote the spectral radius, the numerical radius, and the usual operator norm of T, respectively. Recall that the numerical radius of an operator $T \in \mathbb{B}(\mathbb{H})$ is defined by

$$\omega(T) = \sup_{||x||=1} |\langle Tx, x\rangle|.$$

It is well-known that for $T \in \mathbb{B}(\mathbb{H})$, we have $r(T) \leq \omega(T)$, with equality if T is normal. It is also well-known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $||\cdot||$. For $T \in \mathbb{B}(\mathbb{H})$, we have

$$\frac{||T||}{2} \leq \omega(T) \leq ||T||. \tag{1.1}$$

The first inequality in (1.1) becomes an equality if $T^2 = 0$, and the second inequality becomes an equality if T is normal.

CONTACT Fuad Kittaneh fkitt@ju.edu.jo Department of Mathematics, The University of Jordan, Amman, Jordan.

One of the most important properties of the numerical radius is its weak unitary invariance, that is, for $T \in \mathbb{B}(\mathbb{H})$,

$$\omega(U^*TU) = \omega(T) \tag{1.2}$$

for every unitary operator $U \in \mathbb{B}(\mathbb{H})$. For basic properties of the numerical radius, we refer to [2] and [3].

Kittaneh has shown in [8] that if $T \in \mathbb{B}(\mathbb{H})$, then

$$\omega(T) \leq \frac{1}{2}\left(||T|| + ||T^2||^{\frac{1}{2}}\right).$$

Consequently, if $T^2 = 0$, then

$$\omega(T) = \frac{1}{2}||T||. \tag{1.3}$$

It should be mentioned here that the computation of the numerical radius is an optimization problem. For the relevance of the numerical radius to numerical functional analysis and optimization, we refer to [9, 11], and references therein.

In this article, we present numerical radius inequalities for $n \times n$ operator matrices with a single nonzero row. Then we use these inequalities to establish numerical radius inequalities for arbitrary $n \times n$ operator matrices. Moreover, we give numerical radius inequalities for 3×3 operator matrices that involve the numerical radii of the skew diagonal (or the secondary diagonal) parts of 2×2 operator matrices. Our new numerical radius inequalities for $n \times n$ operator matrices are natural generalizations of some of the numerical radius inequalities for 2×2 operator matrices given in [4, 5, 10], and references therein. The $n \times n$ operator matrices $T = [T_{ij}]$ are regarded as operators on $\bigoplus_{i=1}^{n}\mathbb{H}_i$ (the direct sum of the complex Hilbert spaces $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$), where $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$ for $i, j = 1, 2, \ldots, n$. Here, $\mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$ denotes the space of all bounded linear operators from $\mathbb{H}_j$ to $\mathbb{H}_i$. When $i = j$, we simply write $\mathbb{B}(\mathbb{H}_i)$ for $\mathbb{B}(\mathbb{H}_i, \mathbb{H}_i)$.

2. Main results

To present our results, we need the following lemmas. The first lemma is well-known.

Lemma 2.1. [12] *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(A) = \sup_{\theta \in \mathbb{R}} ||Re(e^{i\theta}A)||.$$

Lemma 2.2. [6, p. 44] *Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, n$. Then*

$$\omega(A) = \frac{1}{2} r([a_{ij} + a_{ji}]).$$

Lemma 2.3. [7] *Let $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$ be Hilbert spaces, and let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$r(T) \le r\big([||T_{ij}||]\big).$$

Lemma 2.4. [1] *Let $T = [T_{ij}]$ be $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(T) \le \omega([t_{ij}]),$$

where

$$t_{ij} = \omega\left(\begin{bmatrix} 0 & T_{ij} \\ T_{ji} & 0 \end{bmatrix}\right) \qquad \text{for} \quad i, j = 1, 2, ..., n.$$

Note that $t_{ii} = \omega(T_{ii})$ for $i = 1, 2, ..., n$ and the matrix $[t_{ij}]$ is real symmetric.

Our first result in this section can be stated as follows.

Theorem 2.5. *Let $A_1 \in \mathbb{B}(\mathbb{H}_1), A_2 \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1), ..., A_n \in \mathbb{B}(\mathbb{H}_n, \mathbb{H}_1)$. Then*

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \le \frac{1}{2}\left(\omega(A_1) + \sqrt{\omega^2(A_1) + \sum_{j=2}^{n} ||A_j||^2}\right).$$

Proof. Applying Lemma 2.4 and the identity (1.3), we have

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$\le \omega\left(\begin{bmatrix} \omega(A_1) & \omega\left(\begin{bmatrix} 0 & A_2 \\ 0 & 0 \end{bmatrix}\right) & \cdots & \omega\left(\begin{bmatrix} 0 & A_n \\ 0 & 0 \end{bmatrix}\right) \\ \omega\left(\begin{bmatrix} 0 & 0 \\ A_2 & 0 \end{bmatrix}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \omega\left(\begin{bmatrix} 0 & 0 \\ A_n & 0 \end{bmatrix}\right) & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$= \omega\left(\begin{bmatrix} \omega(A_1) & \frac{||A_2||}{2} & \cdots & \frac{||A_n||}{2} \\ \frac{||A_2||}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{||A_n||}{2} & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2} r\left(\begin{bmatrix} 2\omega(A_1) & ||A_2|| & \cdots & ||A_n|| \\ ||A_2|| & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ ||A_n|| & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\omega(A_1) + \sqrt{\omega^2(A_1) + \sum_{j=2}^{n} ||A_j||^2}\right).$$

On the basis of Theorem 2.5, we obtain the following numerical radius inequality for arbitrary $n \times n$ operator matrices.

Corollary 2.6. *Let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(T) \le \frac{1}{2}\sum_{i=1}^{n}\left(\omega(T_{ii}) + \sqrt{\omega^2(T_{ii}) + \sum_{j \ne i^{j=1}}^{n} ||T_{ij}||^2}\right).$$

Proof. For $i = 2, \ldots, n$, let U_i be the $n \times n$ permutation operator matrix obtained by interchanging the first and the ith rows of the identity operator matrix. Then U_i is a unitary operator, and so by the triangle inequality and the identity (1.2), we have

$$\omega(T) \le \omega\left(\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$+ \cdots + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}\right)$$

$$= \omega\left(\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \omega\left(U_2^*\begin{bmatrix} T_{22} & T_{21} & \cdots & T_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}U_2\right)$$

$$+ \cdots + \omega\left(U_n^*\begin{bmatrix} T_{nn} & T_{n2} & \cdots & T_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}U_n\right)$$

$$= \omega\left(\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} T_{22} & T_{21} & \cdots & T_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$+ \cdots + \omega\left(\begin{bmatrix} T_{nn} & T_{n2} & \cdots & T_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right).$$

Now, using Theorem 2.5, we have

$$\omega(T) \le \frac{1}{2}\sum_{i=1}^{n}\left(\omega(T_{ii}) + \sqrt{\omega^2(T_{ii}) + \sum_{j\neq i^{j=1}}^{n} ||T_{ij}||^2}\right).$$

Other related results are given as follows.

Theorem 2.7. *Let* $A_1 \in \mathbb{B}(\mathbb{H}_1), A_2 \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1), ..., A_n \in \mathbb{B}(\mathbb{H}_n, \mathbb{H}_1)$. *Then*

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \le \frac{1}{2}\left(||A_1|| + \sqrt{||\sum_{j=1}^{n} A_jA_j^*||^2}\right).$$

Proof. Let $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Then for every $\theta \in \mathbb{R}$, we have

$$
\begin{aligned}
||Re(e^{i\theta}A)|| &= r(Re(e^{i\theta}A)) \\
&= \frac{1}{2} r\left(e^{i\theta}\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + e^{-i\theta}\begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2} r\left(\begin{bmatrix} e^{i\theta}A_1 + e^{-i\theta}A_1^* & e^{i\theta}A_2 & \cdots & e^{i\theta}A_n \\ e^{-i\theta}A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ e^{-i\theta}A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2} r\left(\begin{bmatrix} A_1^* & e^{i\theta}I & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right).
\end{aligned}
$$

It follows by the commutativity property of the spectral radius and Lemma 2.3 that

$$
\begin{aligned}
||Re(e^{i\theta}A)|| &= \frac{1}{2} r\left(\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} A_1^* & e^{i\theta}I & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2} r\left(\begin{bmatrix} e^{-i\theta}A_1^* & I & \cdots & 0 \\ \sum_{j=1}^{n} A_jA_j^* & e^{i\theta}A_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\le \frac{1}{2} r\left(\begin{bmatrix} ||A_1^*|| & 1 & \cdots & 0 \\ \left\|\sum_{j=1}^{n} A_jA_j^*\right\| & ||A_1|| & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(||A_1|| + \sqrt{\left\|\sum_{j=1}^{n} A_jA_j^*\right\|}\right).
\end{aligned}
$$

Now, using Lemma 2.1, we have

$$\omega(A) = \sup_{\theta \in \mathbb{R}} ||Re(e^{i\theta}A)|| \leq \frac{1}{2}\left(\|A_1\| + \sqrt{\left\|\sum_{j=1}^{n} A_j A_j^*\right\|}\right).$$

□

Using Theorem 2.7 and an argument similar to that used in proof of Corollary 2.6, we have the following corollary.

Corollary 2.8. *Let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(T) \leq \frac{1}{2}\sum_{i=1}^{n}\left(||T_{ii}|| + \sqrt{\left\|T_{ii}T_{ii}^* + \sum_{j \neq i^{j=1}}^{n} T_{ij}T_{ij}^*\right\|}\right).$$

Theorem 2.9. *Let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(T) \leq \max\big(\omega(T_{11}), \omega(T_{22}), ..., \omega(T_{nn})\big) + \frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{j \neq i^{j=1}}^{n} T_{ij}T_{ij}^*\right\|}.$$

Proof. First observe that

$$\begin{bmatrix} 0 & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ T_{21} & 0 & T_{23} & \cdots & T_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2 = \cdots =$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ T_{n1} & T_{n2} & \cdots & T_{n(n-1)} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now, by the triangle inequality and the identity (1.3), we have

$$\omega(T) \leq \omega\left(\begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ 0 & T_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_{nn} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$+\omega\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ T_{21} & 0 & T_{23} & \cdots & T_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$+\cdots+\omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ T_{n1} & T_{n2} & \cdots & T_{n(n-1)} & 0 \end{bmatrix}\right)$$

$$= \max\big(\omega(T_{11}), \omega(T_{22}), ..., \omega(T_{nn})\big) + \frac{1}{2}\left\|\begin{bmatrix} 0 & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right\|$$

$$+\frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ T_{21} & 0 & T_{23} & \cdots & T_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right\| + \cdots + \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ T_{n1} & T_{n2} & \cdots & T_{n(n-1)} & 0 \end{bmatrix}\right\|$$

$$= \max\big(\omega(T_{11}), \omega(T_{22}), ..., \omega(T_{nn})\big) + \frac{1}{2}\sqrt{\left\|\sum_{\substack{j=1 \\ j\neq 1}}^{n} T_{1j}T_{1j}^{*}\right\|}$$

$$+\frac{1}{2}\sqrt{\left\|\sum_{\substack{j=1 \\ j\neq 2}}^{n} T_{2j}T_{2j}^{*}\right\|} + \cdots + \frac{1}{2}\sqrt{\left\|\sum_{\substack{j=1 \\ j\neq n}}^{n} T_{nj}T_{nj}^{*}\right\|}$$

$$= \max\big(\omega(T_{11}), \omega(T_{22}), ..., \omega(T_{nn})\big) + \frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{\substack{j=1 \\ j\neq i}}^{n} T_{ij}T_{ij}^{*}\right\|}.$$

□

The proof of the following theorem has been pointed out to us by Amer Abu-Omar. For simplicity, we state it for 3×3 operator matrices.

Theorem 2.10. *Let $T = [T_{ij}]$ be a 3×3 operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(T) \leq \frac{1}{2}\left(\sum_{i=1}^{3} t_{ii} + \sum_{1 \leq i < j \leq 3} \sqrt{\left(\frac{t_{ii} - t_{jj}}{2}\right)^2 + 4t_{ij}^2}\right).$$

Here,

$$t_{ij} = \omega\left(\begin{bmatrix} 0 & T_{ij} \\ T_{ji} & 0 \end{bmatrix}\right) \quad \textit{for} \quad i, j = 1, 2, \ldots, n$$

and

$$t_{ii} = \omega(T_{ii}) \quad \textit{for} \quad i = 1, 2, \ldots, n.$$

Proof. By the triangle inequality and Lemma 2.4, we have

$$\begin{aligned}
&\omega(T) \\
&\leq \omega\left(\begin{bmatrix} \frac{T_{11}}{2} & T_{12} & 0 \\ T_{21} & \frac{T_{22}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} \frac{T_{11}}{2} & 0 & T_{13} \\ 0 & 0 & 0 \\ T_{31} & 0 & \frac{T_{33}}{2} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{T_{22}}{2} & T_{23} \\ 0 & T_{32} & \frac{T_{33}}{2} \end{bmatrix}\right) \\
&\leq \omega\left(\begin{bmatrix} \frac{t_{11}}{2} & t_{12} & 0 \\ t_{21} & \frac{t_{22}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} \frac{t_{11}}{2} & 0 & t_{13} \\ 0 & 0 & 0 \\ t_{31} & 0 & \frac{t_{33}}{2} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{t_{22}}{2} & t_{23} \\ 0 & t_{32} & \frac{t_{33}}{2} \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} \frac{t_{11}}{2} & t_{12} & 0 \\ t_{21} & \frac{t_{22}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} \frac{t_{11}}{2} & 0 & t_{13} \\ 0 & 0 & 0 \\ t_{31} & 0 & \frac{t_{33}}{2} \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{t_{22}}{2} & t_{23} \\ 0 & t_{32} & \frac{t_{33}}{2} \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{3} t_{ii} + \sum_{1 \leq i < j \leq 3} \sqrt{\left(\frac{t_{ii} - t_{jj}}{2}\right)^2 + 4t_{ij}^2}\right) \text{(recall that } t_{ij} = t_{ji}\text{)}.
\end{aligned}$$

□

With the same notations used in Theorem 2.10, we have the following related result.

Theorem 2.11. *Let $T = [T_{ij}]$ be a 3×3 operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(T) \leq \max(t_{11}, t_{23}) + \max(t_{22}, t_{13}) + \max(t_{33}, t_{12}).$$

Here,

$$t_{ij} = \omega\left(\begin{bmatrix} 0 & T_{ij} \\ T_{ji} & 0 \end{bmatrix}\right) \qquad \textit{for} \quad i, j = 1, 2, \ldots, n$$

and

$$t_{ii} = \omega(T_{ii}) \quad \textit{for} \quad i = 1, 2, \ldots, n.$$

Proof. Let $U = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$. Then U is a unitary operator, and so by the triangle inequality and the identity (1.2), we have

$$\omega(T) \leq \omega\left(\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & T_{23} \\ 0 & T_{32} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & T_{12} & 0 \\ T_{21} & 0 & 0 \\ 0 & 0 & T_{33} \end{bmatrix}\right)$$

$$+\omega\left(\begin{bmatrix} 0 & 0 & T_{13} \\ 0 & T_{22} & 0 \\ T_{31} & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & T_{23} \\ 0 & T_{32} & 0 \end{bmatrix}\right)$$

$$+\omega\left(\begin{bmatrix} 0 & T_{12} & 0 \\ T_{21} & 0 & 0 \\ 0 & 0 & T_{33} \end{bmatrix}\right) + \omega\left(U^* \begin{bmatrix} 0 & T_{13} & 0 \\ T_{31} & 0 & 0 \\ 0 & 0 & T_{22} \end{bmatrix} U\right)$$

$$= \omega\left(\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & T_{23} \\ 0 & T_{32} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & T_{12} & 0 \\ T_{21} & 0 & 0 \\ 0 & 0 & T_{33} \end{bmatrix}\right)$$

$$+\omega\left(\begin{bmatrix} 0 & T_{13} & 0 \\ T_{31} & 0 & 0 \\ 0 & 0 & T_{22} \end{bmatrix}\right)$$

$$= \max(t_{11}, t_{23}) + \max(t_{22}, t_{13}) + \max(t_{33}, t_{12}).$$

□

Concerning Theorems 2.10 and 2.11, we note that several bounds for t_{ij} have been given in [4, 5, 10], and references therein. Moreover, when $\mathbb{H}_i = \mathbb{H}_j$, it has been shown in [1] that if T_{ij} and T_{ji} are positive, then $t_{ij} = \frac{||T_{ij} + T_{ji}||}{2}$.

We conclude by remarking that the numerical radius inequalities presented in this article are sharp. Moreover, by employing similar analysis to different partitions of operator matrices, it is possible to obtain further numerical radius inequalities.

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