People’s Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University Center of Barika
Institute of Science
Department of Mathematics

Fuzzy sets and relations 1

Domain: Mathematics and Computer Science
Filed: Mathematics
Speciality: Discrete Mathematics and applications
Level: 1st year of Master
Semestre: 1

Realized by

Dr. Soheyb Milles

Academic year 2023-2024

University Center of Barika
Institute of Science
Department of Mathematics

Course of "Fuzzy sets and relations 1"

Soheyb Milles

Academic year 2023-2024

Please refer to this work as follows :
Soheyb Milles (**2024**). *Fuzzy sets and relations 1*, Department of mathematics, University Center of Barika, Algeria.

Introduction

This handout is the result of reading numerous works, most of which are cited in the list of bibliographical references, and it is directed to first-year students of Master of Mathematics, speciality "discrete mathematics and applications" at University Center of Barika within the framework of the LMD system. I respect in this handout the official program approved in the academic program of semester 1.
This course is structured as follows. In Chapter 1, we provide a basic introduction to crisp sets, fuzzy sets and t-norms and t-conorms. Next, we recall some basic notions related to fuzzy sets. Many of the properties of these concepts will be used in next chapter. In Chapter 2, we investigate the binary relations concepts and their fundamental properties, fuzzy relations, operations and characteristics of fuzzy relations.

At the end of this course you will find a series of unsolved exercises. We will also find at the end of this handout a list of books whose consultation can be a good support, or complement, for reading.

It is certain that the first version of this work can be improved, and that it contains certain remarks, which is why I invite all readers, students or teachers to send me their remarks and comments to my email address : soheyb.milles@cu-barika.dz

Table of contents

1 Fuzzy sets

The purpose of this first chapter is provide a basic introduction to crisp sets, fuzzy sets and t-norms and t-conorms. Next, we recall some basic notions related to fuzzy sets. Many of the properties of these concepts will be used in next chapter.

1.1. Crisp sets

This section contains the basic definitions of crisp sets with several operations.

Définition 1.1. *A set of reference X is a collection of objects, this set con be defined by*

(i) Writing of all its elements, whose elements are $a_1, a_2, \cdots, a_n$, and we write, $X = \{a_1, a_2, \cdots, a_n\}$.

(ii) A property or properties are satisfied by its elements, and we write, $A = \{x \mid P(x)\}$. Where the symbol "$|$" denotes the sentence "such that" and $P(x)$ a proposition of the form "x" has a property.

(iii) A function called characteristic function $\mathcal{X}_A(x)$ which takes the value 0 for the elements that do not belong to A and the value 1 for those that belong to A :

$$\chi_A : X \longrightarrow \{0,1\}$$
$$x \longmapsto \begin{cases} 0 & if \quad x \notin A; \\ 1 & if \quad x \in A. \end{cases}$$

Définition 1.2. ***(Operations on crisp sets)*** *Let X be a set, let A and B be two subsets on X.*

(i) Inclusion : $A \subset B$ if $(x \in A) \Rightarrow (x \in B)$, i.e.,$\mathcal{X}_A(x) \leq \mathcal{X}_B(x)$, for any $x \in X$;

(ii) *Equality :* $A = B$ *if* $A \subseteq B$ *and* $B \subseteq A$*, i.e.,* $(\mathcal{X}_A(x) = \mathcal{X}_B(x))$*, for any* $x \in X$*;*

(iii) *Complement :* $A^c = \{x \in X \mid x \notin A\}$*, i.e.,* $\mathcal{X}_{A^c}(x) = 1 - \mathcal{X}_A(x)$*;*

(iv) *Intersection :* $A \cap B = \{x \in X \mid x \in A$ *and* $x \in B\}$*, i.e.,*$\mathcal{X}_{A\cap B}(x) = min(\mathcal{X}_A(x), \mathcal{X}_B(x))$*;*

(v) *Union :* $A \cup B = \{x \in X \mid x \in A$ *or* $x \in B\}$*, i.e.,*$\mathcal{X}_{A\cup B}(x) = max(\mathcal{X}_A(x), \mathcal{X}_B(x))$*;*

(vi) *Relative complement :* $A \backslash B = A - B = A \cap B^c = \{x \in X \mid x \in A$ *and* $x \notin B\}$*, i.e.,* $\mathcal{X}_{A-B}(x) = \mathcal{X}_{A\cap B^c}(x) = min(\mathcal{X}_A(x), \mathcal{X}_{B^c}(x))$*.*

Exemple 1.1. *Let* $X = \{x, y, z, t, w\}$ *be a set, let* A *and* B *be two subsets of* X *such that* $A = \{x, y, w\}$ *and* $B = \{x, y, z\}$*. Then,*

$$A^c = \{z, t\};$$

$$B^c = \{t, w\};$$

$$A \cap B = \{x, y\};$$

$$A \cup B = \{x, y, z, w\};$$

$$A \backslash B = \{w\};$$

$$B \backslash A = \{z\}.$$

Exemple 1.2. *Let* $X = [36, 42]$ *the universe of speech which expresses the degree of temperature of a human body, a person with hepatitis generally presents the following symptoms :*
(i) The person has a high fever ;
(ii) His skin is yellow in color ;
(iii) He has nausea.
We will now study the first symptom or the property having a high fever which is indicated by the degree of temperature in the classic case (see ***Figure 1.1****)we define of temperature in the classic set* A *of* X *associated with the property (having a high fever) by :*

$$\mathcal{X}_A(x) = \begin{cases} 1, & if \quad x \geq 39 \\ 0, & otherwise. \end{cases}$$

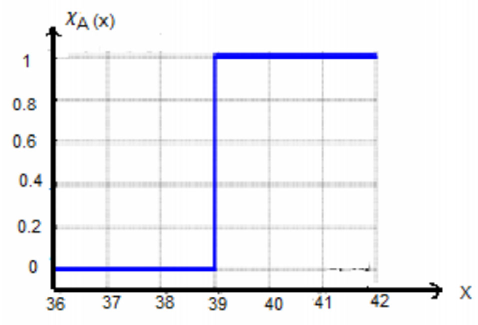

Figure 1.1

i.e., people have a temperature over 39 are systematically have a high fever therefore they reach hepatitis, without this diagnosis being logical.

1.2. Fuzzy sets

This section contains the basic definitions and properties of fuzzy sets and several operations of fuzzy sets. The notion of fuzzy set was introduced in 1965 by Lotfi A. Zadeh in the paper [17].

Définition 1.3. [17] *Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for $x \in X$.*

Notation 1.1. *Let X be a nonempty set. The set of all fuzzy subsets of X will be denoted by $F(X)$.*

Exemple 1.3. *Let $X = \{a, b, c\}$ be a set. $A_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ and*
$A_2 = \{(a, 0.0), (b, 0.9), (c, 0.7)\}$ are two fuzzy subsets on X.

Exemple 1.4. *In same **Example1.2**, in this fuzzy case the (**Figure 1.2**) shows a possible diagnosis. We also define the fuzzy subset B of X*

which associated with the same property of A (to have a strong fever) as

$$\mu_B(x) = \begin{cases} 0, & if \quad 36 \leq x \leq 37, \\ \frac{26}{110}x - \frac{1037}{110}, & if \quad 37 < x < 41, \\ 1, & if \quad 41 < x < 42. \end{cases}$$

i.e., a person has a temperature X of $[36, 42]$

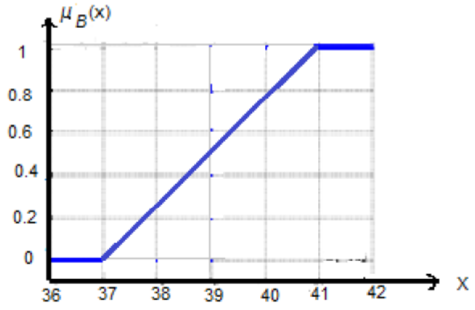

Figure 1.2 – Trapezoid diagram.

If $\mu_B(x) = 1$, so the person with hepatitis.
If $\mu_B(x) = \frac{29}{110}x - \frac{1037}{110}$, so the person the patient with a degree of hepatitis.
If $\mu_B(x) = 0$, so the person does not have hepatitis.

1.2.1. Fuzzy sets operations

For two fuzzy sets A and B on a set X, several operations are defined in the following way(see [17])

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$, for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x) \rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x) \rangle \mid x \in X\}$;

(v) $\bar{A} = \{\langle x, 1-\mu_A(x)\rangle \ | \ x \in X\}$.

Exemple 1.5. *If we consider the fuzzy sets*

$$A_1(x) = \begin{cases} 1, & if \ \ 40 \leq x < 50, \\ 1 - \frac{x-50}{10}, & if \ \ 50 \leq x < 60, \\ 0, & if \ \ 60 \leq x \leq 100. \end{cases}$$

$$A_2(x) = \begin{cases} 0, & if \ \ 40 \leq x < 50, \\ \frac{x-50}{10}, & if \ \ 50 \leq x < 60, \\ 1 - \frac{x-60}{10}, & if \ \ 60 \leq x < 70, \\ 0, & if \ \ 70 \leq x \leq 100. \end{cases}$$

Then their union is

$$(A_1 \cup A_2)(x) = \begin{cases} 1, & if \ \ 40 \leq x < 50, \\ 1 - \frac{x-50}{10}, & if \ \ 50 \leq x < 55, \\ \frac{x-50}{10}, & if \ \ 55 \leq x \leq 60, \\ 1 - \frac{x-60}{10}, & if \ \ 60 \leq x \leq 70, \\ 0, & if \ \ 70 \leq x \leq 100. \end{cases}$$

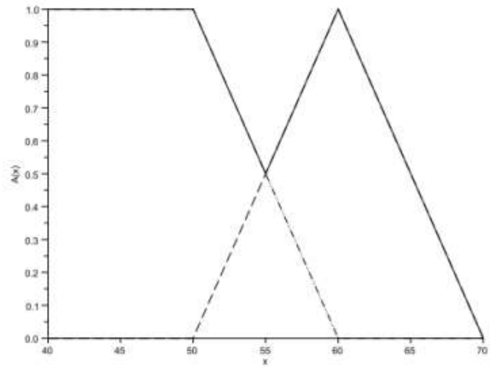

Figure 1.3 – Fuzzy Union.

The intersection can be expressed as

$$(A_1 \cap A_2)(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 50 \leq x < 55, \\ 1 - \frac{x-50}{10}, & if \quad 55 \leq x < 60, \\ 0, & if \quad 60 < x \leq 100. \end{cases}$$

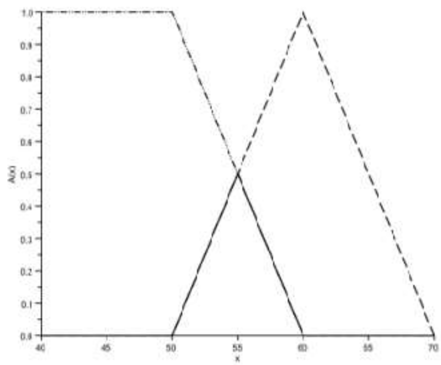

FIGURE 1.4 – Fuzzy Intersection.

The complement of A_1 can be written

$$\overline{A_1}(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 40 \leq x < 60, \\ 1, & if \quad 60 \leq x \leq 100. \end{cases}$$

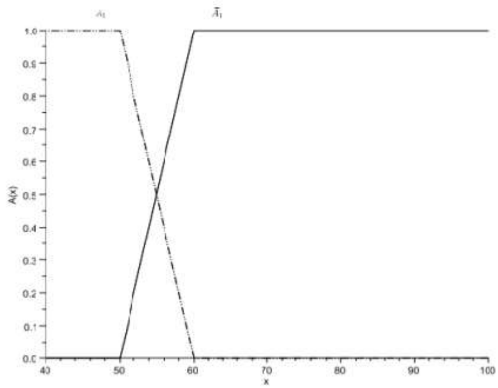

FIGURE 1.5 – The complement of a fuzzy set.

Exemple 1.6. *Let* $X = \mathbb{R}$ *and let* A *be the set of reals greater than* 10 *and* B *the set of reals close to* 1 *are characterized respectively by its membership functions*

$$\mu_A(x) = \begin{cases} 0, & if \quad x \leq 10, \\ (1 + (x - 10)^{-2})^{-1}, & if \quad x > 10, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & if \quad x \leq 10, \\ (1 + (x - 10)^{4})^{-1}, & if \quad x > 10. \end{cases}$$

So, we get $A \cap B$ *set of reals greater than* 10 *and close to* 11 *given by its membership function*

$$\mu_{A\cap B}(x) = \begin{cases} 0, & if \quad x \leq 10, \\ min[(1 + (x - 10)^{-2})^{-1}, (1 + (x - 10)^{4})^{-1}], & if \quad x > 10. \end{cases}$$

And $A \cup B$ *the set of real numbers greater than 10 or close to 11 given by its membership function*

$$\mu_{A\cup B}(x) = max[(1 + (x - 10)^{-2})^{-1}, (1 + (x - 10)^{4})^{-1}], x \in X.$$

1.2.2. Characteristics of fuzzy sets

In this subsection, we present some characteristics of fuzzy sets.

Définition 1.4. ***(The support of a fuzzy sets)*** [15] *Let A be a fuzzy set on a set X. The support of A is the crisp subset on X given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Définition 1.5. ***(The kernel of a fuzzy sets)*** [15] *Let A be a fuzzy set on a set X. The kernel of A is the crisp subset on X given by*

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Définition 1.6. ***(The highest of a fuzzy sets)*** [15] *Let A be a fuzzy set on a set X. The height of A is the highest value taken by its membership function given by*

$$H(A) = sup_{x \in X} \mu_A(x).$$

Définition 1.7. ***(The cardinality of a fuzzy sets)*** [8] *The cardinality of the fuzzy subset A of X, noted $\mid A \mid$, when X is finite, is defined by*

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Exemple 1.7. *Let $X = \{a, b, c\}$ be a set. $A_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ and $A_2 = \{(a, 0.0), (b, 0.9), (c, 1.0)\}$ are two fuzzy subsets on X. Then, $Supp(A_1) = \{a, b, c\}$ and $Supp(A_2) = \{b, c\}$.*
$Ker(A_1) = \{b\}$ and $Ker(A_2) = \{c\}$. $H(A_1) = 1$ and $H(A_2) = 1$. $\mid A_1 \mid = 2$ and $\mid A_2 \mid = 1.9$.

Exemple 1.8. *Let $X = [0, 1]$ with $\alpha, \beta \in \mathbb{R}$ and let $a, b \in \mathbb{R}$. We define the fuzzy set A on X by*

$$\mu_A(x) = \begin{cases} 0, \quad if \quad x < a - \alpha \quad or \quad b + \beta < x, \\ 1, \quad if \quad a < x < b, \\ 1 + (\frac{x-a}{\alpha}), \quad if \quad a - \alpha < x < a, \\ 1 - (\frac{b-x}{\beta}), \quad if \quad b < x < b + \beta. \end{cases}$$

Then, $Ker(A) = [0, 1]$*,* $Supp(A) = [a - \alpha, b + \beta]$ *and* $H(A) = 1$*.*

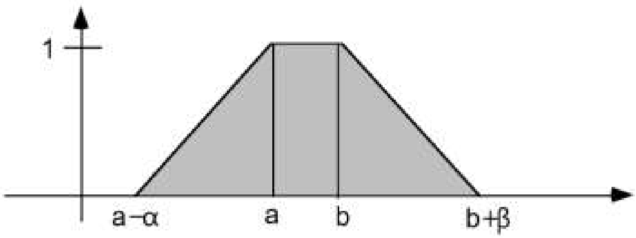

FIGURE 1.6

Exemple 1.9. *Let* B *the fuzzy subset given by* ***Figure 1.2*** *on the set* $X = [36, 42]$*. Then,*
$Supp(A) =]37, 42]$*,* $H(A) = 1$*,* $Ker(A) = [41, 42]$*,* $\mid A \mid$ *is infinite.*

Définition 1.8. ***(The*** α***-cut of a fuzzy set)*** [15] *Let* A *be a fuzzy set on a set* X*. The* α*-cut of* A *is the crisp subset*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \text{ where } \alpha \in [0, 1].$$

Particular cases :

1. *If* $\alpha = 0$*, then* $A_0 = X$*.*
2. *If* $\alpha = 1$*, then* $A_1 = Ker(A)$*.*

Exemple 1.10. *Let* $X = \{1, 2, 3, ..., 10\}$*, and* A *be a fuzzy subset of* X *given by* $A = \{< 1; 0.2 >, < 2; 0.5 >, < 3; 0.8 >, < 4; 1 >, < 5; 0.7 >, < 6; 0.3 >, < 7; 0 >, < 8; 0 >, < 9; 0 >, < 10; 0 >\}$*. Then, the* α*-cut of* A *is given by :*

$A_0 = \{x \in X, A(x) > 0\} = X;$
$A_{0.1} = \{x \in X, A(x) \geq 0.1\} = \{1, 2, 3, 4, 5, 6\};$
$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{1, 2, 3, 4, 5, 6\};$
$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{2, 3, 4, 5, 6\};$
$A_{0.4} = \{x \in X, A(x) \geq 0.4\} = \{2, 3, 4, 5\};$
$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{2, 3, 4, 5\};$
$A_{0.6} = \{x \in X, A(x) \geq 0.6\} = \{3, 4, 5\};$
$A_{0.7} = \{x \in X, A(x) \geq 0.7\} = \{3, 4, 5\};$
$A_{0.8} = \{x \in X, A(x) \geq 0.8\} = \{3, 4\};$
$A_{0.9} = \{x \in X, A(x) \geq 0.9\} = \{4\};$
$A_1 = \{x \in X, A(x) \geq 1\} = \{4\}.$

1.2.3. Projection and cartesian product on fuzzy sets

Définition 1.9. ***(Cartesian product on fuzzy set)*** [7] *Let the fuzzy subsets $A_1, A_2, \cdots, A_n$ respectively defined on $X_1, X_2, \cdots, X_n$, we define their cartesian product $A = A_1 \times A_2 \times \cdots \times A_n$, as a fuzzy subset of $X = X_1, X_2, \cdots, X_n$ with a membership function defined for any $x = (x_1, x_2, \cdots, x_n) \in X$ by :*

$$\mu_A(x) = min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)].$$

Exemple 1.11. *Let X_1 be a set of animals $X_1 = \{cat, cheetah, tiger\}$ and X_2 be a set of country choices by temperature $X_2 = \{hot, cold\}$. The fuzzy subset A_1 represents the choices of an individual that the animals would like to own and the fuzzy subset A_2 represents its choices to the type of country in which the animal would like to live such as $A_1 = \{\langle cat, 0.5\rangle, \langle cheetah, 0.8\rangle, \langle tiger, 0.3\rangle\}, A_2 = \{\langle hot, 0.9\rangle, \langle cold, 0.1\rangle\}$. We get $A_1 \times A_2 = \{\langle (cat, hot), 0.5\rangle, \langle (cat, cold), 0.1\rangle, \langle (cheetah, hot), 0.8\rangle, \langle (cheetah, cold), 0.1\rangle, \langle (tiger, hot), 0.3\rangle, \langle (tiger, cold), 0.1\rangle\}$.*
Let A be a fuzzy subset defined on a universe $X_1 \times X_2$ cartesian product of two reference sets X_1 and X_2.

Définition 1.10. ***(Projection on fuzzy set)*** [7] *The projection on X_1 of the fuzzy set A of $X_1 \times X_2$ is the fuzzy set $Proj_{X_1}(A)$ of X_1,* *whose membership function is defined by :*

$$\forall x_1 \in X_1, \mu_{Proj_{X_1}(A)}(x_1) = sup_{x_2 \in X_2} \mu_A(x_1, x_2).$$

We defined in a similar way the projection of A *on* X_2*.*

Exemple 1.12. *Let* $X = X_1 \times X_2$ *the reference set such that* X_1 *and* X_2 *two sets which are defined in* ***Example 1.11****, we consider* $A_1 \times A_2 = A$ *given by*
$A = \{\langle(cat, hot), 0.5\rangle, \langle(cat, cold), 0.1\rangle, \langle(cheetah, hot), 0.8\rangle,$
$\langle(cheetah, cold), 0.1\rangle, \langle(tiger, hot), 0.3\rangle, \langle(tiger, cold), 0.1\rangle\}$*. Then, we get* $Proj_{X_1}(A) = \{\langle cat, max(0.5, 0.1)\rangle, \langle cheetah, max(0.8, 0.1)\rangle$
$, \langle tiger, max(0.3, 0.1)\rangle\} = \{\langle cat, 0.5\rangle, \langle cheetah, 0.8\rangle, \langle tiger, 0.3\rangle\}$.
$Proj_{X_2}(A) = \{\langle hot, max(0.5, 0.8, 0.3)\rangle, \langle cold, max(0.1, 0.1, 0.1)\rangle\}$
$= \{\langle hot, 0.8\rangle, \langle cold, 0.1\rangle\}$.

In the following theorem, we study the decomposition theorem.

Théorème 1.1. ***(Decomposition theorem)*** [7] *Any fuzzy subset* A *of the reference set* X *is defined from its* α*-cups for any element* x *of* X*.*

$$\mu_A(x) = sup_{\alpha \in]0,1]}(\alpha.\mathcal{X}_{A_\alpha}(x)).$$

$\mathcal{X}_{A_\alpha}$ *is the characteristic function of* A_α*.*

Démonstration. Let the characteristic function

$$\mathcal{X}_{A_\alpha}(x) = \begin{cases} 1, & if \;\; \mu_A(x) \geq \alpha \\ 0, & if \;\; otherwise, \end{cases}$$

by multiplying each member by a real number α, we get :

$$\alpha\mathcal{X}_{A_\alpha}(x) = \begin{cases} \alpha, & if \;\; \mu_A(x) \geq \alpha \\ 0, & if \;\; otherwise. \end{cases}$$

By introducing the "sup" operator in each member, we have :
$sup_{\alpha \in]0,1]} \alpha\mathcal{X}_{A_\alpha}(x) = sup_{\alpha \in]0,1]}\{\mu_A(x) \geq \alpha\} \Rightarrow sup_{\alpha \in]0,1]} \alpha\mathcal{X}_{A_\alpha}(x) = sup_{\alpha \in]0,1]}\{\alpha \leq \mu_A(x)\}$. Or (characterization of the upper bound in

$\mathbb{R}$) : $q = sup(A)$ if and only if $\forall q \in A, x \leq q$ (q is an upper bound of A). Which makes it possible to establish that : $\mu_A(x) = sup_{\alpha\in]0,1]}(\alpha.\mathcal{X}_{A_\alpha}(x))$. $\square$

Exemple 1.13. *Let* $X = \{1, 2, \cdots, 10\}$
and $A = \{\langle 1, 0.2\rangle, \langle 2, 0.5\rangle, \langle 3, 0.8\rangle, \langle 4, 1\rangle, \langle 5, 0.7\rangle, \langle 6, 0.3\rangle\}$.

We have

$$A_1 = \{x \in X|\mu_A(x) \geq 1\} = \{4\};$$

$$A_{0.8} = \{x \in X|\mu_A(x) \geq 0.8\} = \{3, 4\};$$

$$A_{0.7} = \{x \in X|\mu_A(x) \geq 1\} = \{3, 4, 5\};$$

$$A_{0.5} = \{x \in X|\mu_A(x) \geq 1\} = \{2, 3, 4, 5\};$$

$$A_{0.3} = \{x \in X|\mu_A(x) \geq 1\} = \{2, 3, 4, 5, 6\};$$

$$A_{0.2} = \{x \in X|\mu_A(x) \geq 1\} = \{1, 2, 3, 4, 5, 6\}.$$

So, we get
$\mu_A(1) = max(1 \times 0, \cdots, 0.2 \times 1, 0.1 \times 1, 0 \times 1) = 0.2;$
$\mu_A(2) = max(1 \times 0, \cdots, 0.5 \times 1, 0.4 \times 1, 0 \times 1) = 0.5;$
$\mu_A(3) = max(1 \times 0, 0.9 \times 0, 0.8 \times 1, \cdots, 0 \times 1) = 0.8;$
$\mu_A(4) = max(1 \times 1, \cdots, 0 \times 1) = 1;$
$\mu_A(5) = max(1 \times 0, \cdots, 0.7 \times 1, \cdots, 0 \times 1) = 0.7;$
$\mu_A(6) = max(1 \times 0, \cdots, 0.3 \times 1, \cdots, 0 \times 1) = 0.3.$
Which provides the set A.

1.3. T-norms and t-conorms

The history of triangular-norms (t-norms) started with Menger [6]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements.

1.3.1. T-norms

Définition 1.11. [13] *A t-norm T on $[0,1]$ is a function $T : [0,1]^2 \to [0,1]$ satisfies the following four axioms :*

(T1) Commutativity : $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$.

(T2) Associativity : $(\forall x, y, z \in [0,1])(T(x, T(y,z)) = T(T(x,y), z))$.

(T3) Monotonicity : $(\forall x, y, z \in [0,1])(x \leq y \Rightarrow T(x,z) \leq T(y,z))$.

(T4) Boundary condition : $(\forall x \in [0,1])(T(x,1) = x)$.

Conditions $(T4)$ and $(T3)$ imply that for any t-norm T it holds that $T(x,y) \leq x, T(x,y) \leq y, T(x,y) \leq Min(x,y)$ and $T(x,0) = 0$.

Exemple 1.14. *The following four operations are the most common t-norms :*

(T5) Minimum : $T_M(x,y) = min\{x,y\}$.

(T6) Product : $T_P(x,y) = x \cdot y$.

(T7) Lukasiewicz : $T_L(x,y) = max\{x+y-1, 0\}$.

(T8) Drastic product :

$$T_D(x,y) = \begin{cases} x, & if \quad y = 1 \\ y, & if \quad x = 1 \\ 0, & if \quad x, y < 1. \end{cases}$$

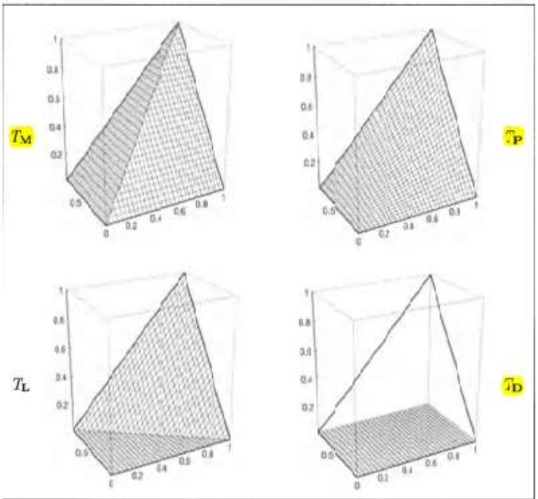

FIGURE 1.7 – 3D plots of the four basic t-norm.

Let T be a t-norm on $[0,1]$. An element $\alpha \in]0,1[$ is called a zero divisor of T if there exists some $b > 0$ such that $T(a,b) = 0$. An element $\alpha \in [0,1]$ is called an idempotent element of T if $T(a,a) = a$. T is called Archimedean if $T(x,x) < x$, for any $x \in [0,1]$. Each $\alpha \in [a,b]$ is an idempotent element of the Minimum t-norm T_M(Actually T_M is the only t-norm whose set of idempotent is equal $[0,1]$), T_M has no zero divisor. Each $\alpha \in]0,1[$ is a zero divisor of the Lukasiewicz t-norm T_L as well of the Drastic product t-norm T_D. For two t-norms T_1 and T_2 on $[0,1]$, we define :

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0,1])(T_1(x,y) \leq T_2(x,y)).$$

Let T_1 and T_2 be two t-norms. If $T_1 \leq T_2$, then T_1 is called weaker than T_2 (or equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t-norm, and T_M is the strongest t-norm, i.e., for any t-norm it holds : $(T9)$ $T_D \leq T \leq T_M$. Since, $T_L \leq T_P$, it obviously holds : $(T10)$ $T_D \leq T_L \leq T_P \leq T_M$.

Exemple 1.15. *1.* $T_0(x,y) = \begin{cases} 0, & if \quad (x,y) \in [0,1[^2 \\ min(x,y), & otherwise; \end{cases}$

2. $T_1(x,y) = max(x+y-1, 0)$;

3. $T_{1.5}(x,y) = \frac{xy}{2-x-y+xy}$;

4. $T_2(x,y) = xy$;

5. $T_{2.5}(x,y) = \frac{xy}{x+y-xy}$;

6. $T_3(x,y) = min(x,y)$.

We have $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Démonstration. 1. $T_0(x,y) = \begin{cases} 0, & if \quad (x,y) \in [0,1[^2; \\ min(x,y), & otherwise. \end{cases}$

If $(x,y) \in [0,1[^2$ then, $T_0 \leq T_1$.

If $(x,y) \notin [0,1[^2$, i.e., $(x,y) \in \{1\} \times [0,1]$ or $[0,1] \times \{1\}$.

If $(x,y) \in \{1\} \times [0,1] : T_0(x,y) = T_0(1,y) = y$ and $T_1(x,y) = T_1(1,y) = y$ then, $T_0 \leq T_1$.

If $(x,y) \in [0,1] \times \{1\} : T_0(x,y) = T_0(x,1) = x$ and $T_1(x,y) = T_1(x,1) = x$ then, $T_0 \leq T_1$. So, $T_0(x,y) \leq T_1(x,y)$.
Then, $T_0 \leq T_1$.

2. $T_1(x,y) = max(x+y-1, 0)$ there are two cases :

(1) $x+y-1 \leq 0 \Rightarrow T_1(x,y) = max(x+y-1,0) = 0 \leq T_{1.5}(x,y)$;

(2) $x+y-1 > 0 \Rightarrow T_1(x,y) = max(x+y-1,0) = x+y-1$.

$$\begin{aligned} T_{1.5}(x,y) - T_1(x,y) &= \frac{xy}{2-x-y+xy} - (x+y-1) \\ &= \frac{(xy - (x+y-1)(2-(x+y)+xy))}{2-x-y+xy}. \end{aligned}$$

Since $(2-x-y+xy) > 0$, it is suffices to determine the sign of the numinator $[xy + (x+y-1)(x+y-xy-2)]$
$(x+y-1)(x+y-xy-2) + xy$
$= (x+y-1)((x+y-1) - (xy+1)) + xy$
$= (x+y-1)^2 - (x+y-1)(xy+1) + xy$
$= (x+y-1)^2 - x^2y - xy^2 + 2xy$

$= (x+y-1)^2 + (xy - x^2y) + (xy - xy^2) \geq 0.$
Therefore, $T_{1.5}(x,y) - T_1(x,y) \geq 0.$
Then, $T_1(x,y) \leq T_{1.5}(x,y).$

(3) $T_2(x,y) = xy.$

$$\begin{aligned} T_{1.5}(x,y) - T_2(x,y) &= \frac{xy}{2-x-y+xy} - xy \\ &= \frac{xy - xy(2-(x+y)+xy)}{2-x-y+xy}. \end{aligned}$$

Since $2-(x+y)+xy > 0$, thus it is enough to determine the sign of the numinator

$$\begin{aligned} xy - xy(2-(x+y)+xy) &= xy + xy(x+y-xy-2) \\ &= xy(x+y-xy-1) \\ &= xy((x-1)+y(1-x)) \\ &= xy(x-1)(1-y) \leq 0. \end{aligned}$$

Thus, $T_{1.5}(x,y) - T_2(x,y) \leq 0$. Then, $T_{1.5}(x,y) \leq T_2(x,y)$.

(4) $T_{2.5}(x,y) = \frac{xy}{x+y-xy}$.

$$\begin{aligned} T_2(x,y) - T_{2.5}(x,y) &= xy - \frac{xy}{x+y-xy} \\ &= \frac{xy(x+y-xy) - xy}{x+y-xy} \\ &= \frac{xy(x+y-xy-1)}{x+y-xy}. \end{aligned}$$

The denominator is positive $(x+y-xy > 0)$, the numerator sign should be studied

$$\begin{aligned} xy(x+y-xy-1) &= xy(x(1-y)+(y-1)) \\ &= xy((1-y)(1-x)) \leq 0. \end{aligned}$$

Thus, $T_2(x,y) - T_{2.5}(x,y) \leq 0$. Then, $T_2(x,y) \leq T_{2.5}(x,y)$.

3. Finally,

$$T_{2.5}(x,y) - T_3(x,y) = \begin{cases} \frac{xy}{x+y-xy} - x, & if \quad x \leq y; \\ \frac{xy}{x+y-xy} - y, & otherwise. \end{cases}$$

If $x \leq y : T_{2.5}(x,y) \leq min(x,y)$.
If $x > y : T_{2.5}(x,y) \leq min(x,y)$.
Thus, for all $(x,y) \in [0,1[^2 \;\; T_{2.5}(x,y) \leq T_3(x,y)$.
Consequently,

$$T_0(x,y) \leq T_1(x,y) \leq T_{1.5}(x,y) \leq T_2(x,y) \leq T_{2.5}(x,y) \leq T_3(x,y).$$

$\square$

Définition 1.12. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $([0,1], *)$, *is a topological monoid with unit* 1 *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ $(a,b,c,d \in [0,1])$.

1.3.2. T-conorms

Définition 1.13. [13] *A t-conorm is a function* $S : [0,1]^2 \to [0,1]$ *that for any* $x,y,z \in [0,1]$ *satisfies* $(T1) - (T3)$ *of Definition 1.11 and the following boundary condition* $S(x,0) = S(0,x) = x$, $S(x,1) = S(1,x) = 0$.

Remark 1.1. *Given a t-norm* T, *we find the associated dual t-conorm* S *by* $S(x,y) = 1 - T(1-x, 1-y)$. *The dual t-conorms w.r.t.* T_M, T_P, T_L *and* T_D *are given by :*

(S1) Maximum : $S_M(x,y) = max\{x,y\}$.

(S2) Probabilistic sum : $S_P(x,y) = x + y - x \cdot y$.

(T7) Lukasiewicz : $S_L(x,y) = min\{x+y, 1\}$.

(T8) Drastic sum :

$$S_D(x,y) = \begin{cases} 1, & if \quad (x,y) \in [0,1]^2 \\ max\{x,y\}, & otherwise. \end{cases}$$

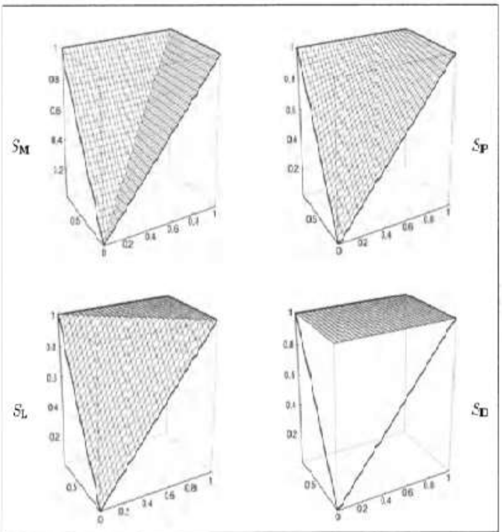

Figure 1.8 – 3D plots of the four basic t-conorm.

Définition 1.14. ***(Duality between operation)*** [7] *A t-norm T and a t-conorm S are said to be dual for the strict negation n if they satisfy the following formulas for all $x, y \in [0, 1]$:*
$S(x, y) = N(T(N(x), N(y)))$;
$T(x, y) = N(S(N(x), N(y)))$.

Exemple 1.16. *Let $X = \{a, b, c\}$, let A and B be two fuzzy subsets of X such that*
$A = \{\langle a, 0.2\rangle, \langle b, 0.4\rangle, \langle c, 0.8\rangle\}$, $B = \{\langle a, 0.9\rangle, \langle b, 0.1\rangle, \langle c, 0.5\rangle\}$. *We can use the operators of Lukasiewicz to define the union and the intersection by*

1. $\mu_{A\cap_T B}(x) = T(\mu_A, \mu_B) = max(\mu_A(x)+\mu_B(x)-1, 0),\ for\ any\ x \in X$;
2. $\mu_{A\cup_S B}(x) = S(\mu_A, \mu_B) = min(\mu_A(x)+\mu_B(x), 1),\ for\ any\ x \in X$.

Then, we get

1. $A \cap_T B = \{\langle a, 0.1\rangle, \langle b, 0\rangle, \langle c, 0.3\rangle\}$;
2. $A \cup_S B = \{\langle a, 1\rangle, \langle b, 0.5\rangle, \langle c, 1\rangle\}$.

2 Fuzzy relations

The purpose of this second chapter is to provide a basic introduction to the binary relations, fuzzy relations, operations and characteristics of fuzzy relations.

2.1. Binary relations

A binary relation on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements, denoted by $x \nparallel y$, if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \parallel_R y$, or simply $x \parallel y$ when no confusion can occur. We denote by R^c the complement of the relation R on X, i.e., for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R on X, i.e., for any $x, y \in X$, xR^ty denotes the fact that yRx. We denote by R^d the dual of the relation R on X, i.e., for any $x, y \in X$, xR^dy denotes the fact that yR^cx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. If $R \cap S = \emptyset$, then R and S are called disjoint relations. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$. For any $n \in \mathbb{N}^*$, the n-th power relation R^n of R is recursively defined as follows :

$$(R^1 = R) \wedge (\forall n \geqslant 1)(R^{n+1} = R^n \circ R)\,.$$

Définition 2.1. *A binary relation R on a set X is called :*

(i) reflexive, if, for any $x \in X$, it holds that xRx ;

(ii) irreflexive, if, for any $x \in X$, it holds hat xR^cx ;

(iii) symmetric, if, for any $x, y \in X$, it holds that xRy implies that yRx ;

(iv) antisymmetric, if, for any $x, y \in X$, it holds that xRy and yRx imply that $x = y$;

(v) asymmetric, if, for any $x, y \in X$, it holds that xRy implies that yR^cx ;

(vi) transitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xRz ;

(vi) antitransitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xR^cz ;

(vii) complete, if, for any $x, y \in X$, either xRy or yRx holds.

Définition 2.2. *A binary relation R on a set X is called :*

(i) a pseudo-order relation if it is reflexive and antisymmetric ;

(ii) an order relation if it is reflexive, antisymmetric and transitive ;

(iii) a strict order if it is irreflexive and transitive ;

(iv) a total order relation if it is reflexive, antisymmetric, transitive and complete ;

((v)) a tolerance relation if it is reflexive and symmetric ;

(vi) an equivalence relation if it is reflexive, symmetric and transitive.

A set X equipped with an order relation $\leqslant$ is called a partially ordered set (poset, for short), denoted $(X, \leqslant)$. Further, $\{x, y\}^u$ denotes the set of all upper bounds of x and y, while $\{x, y\}^l$ denotes the set of all lower bounds of x and y, i.e., $\{x, y\}^u = \{z \in X \mid x \leqslant z \ \wedge \ y \leqslant z\}$ and $\{x, y\}^l = \{z \in X \mid z \leqslant x \ \wedge \ z \leqslant y\}$.

To any order relation $\leqslant$ corresponds a strict order relation $<$ (its strict part or irreflexive kernel) : $x < y$ if $x \leqslant y$ and $x \neq y$. Conversely, to any strict order relation $<$ corresponds an order relation $\leqslant$ (its reflexive closure) : $x \leqslant y$ if $x < y$ or $x = y$. A set X equipped with an order relation $\leqslant$ is called a partially ordered set (poset, for short), denoted by $(X, \leqslant)$. To any order relation $\leqslant$ corresponds a strict order relation $<$ (its strict part or irreflexive kernel) : $x < y$ if $x \leqslant y$ and $x \neq y$. Conversely, to any strict order relation $<$ corresponds an order relation $\leqslant$ (its reflexive closure) :$x \leqslant y$ if $x < y$ or $x = y$.

For any equivalence relation R on a set X, the tolerance/equivalence class of an element $x \in X$ is given by $[x]_R = \{y \in X \mid xRy\}$.

2.2. Fuzzy relations

In this section, we recall the basic definitions and properties of fuzzy relations and several operations on fuzzy relations. The notion of fuzzy relation was introduced in 1971 by Lotfi A.Zadeh in the paper [18].

Définition 2.3. [18] *Let X and Y be two nonempty sets. A binary fuzzy relation R from X to Y, is a fuzzy subset of $X \times Y$ characterized by a membership function R which associates with each pair (x, y) its grade of membership $R(x, y)$ in the interval* $[0, 1]$.

Exemple 2.1. *Let $X = \{a, b, c, d, e\}$. Then the following table represents a fuzzy relation R defined on X.*

$R(.,.)$	a	b	c	d	e
a	1	0	0	0.55	0.40
b	0.34	0.12	0	0.35	0.45
c	0	0	1	0	0.70
d	0.99	0.78	0.22	1	0
e	0	0.78	0	0	0.34

Exemple 2.2. *The fuzzy relation R " x approximately equal to 1" can be defined on $\mathbb{R} \times \mathbb{R}$ by the membership function :*

$$R(x, 1) = \frac{1}{1 + (x - 1)^2}$$

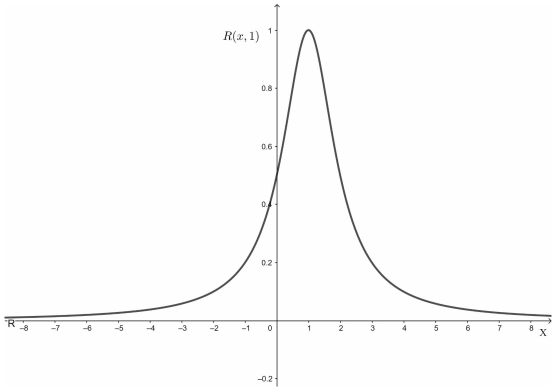

Figure 2.1 – Fuzzy relation.

2.3. Operations on fuzzy relations

Définition 2.4. [17, 18] *Let R, P be two fuzzy relations. R is said to be contained in P (or P contains R), denoted by $R \subseteq P$, if for any $(x,y) \in X \times Y$ it holds that $R(x,y) \leq P(x,y)$. The transpose(or the inverse) R^t of R is the fuzzy relation from Y to X defined by $R^t = \{\langle (x,y), R^t(x,y) \rangle \mid (x,y) \in X \times Y\}$, where $R^t(x,y) = R(y,x)$ for any $(x,y) \in X \times Y$.*
The intersection of two fuzzy relations R and P is defined as $R \cap P = \{\langle (x,y), min(R(x,y), P(x,y)) \rangle \mid (x,y) \in X \times Y\}$.
The union of two fuzzy relations R and P is defined as $R \cup P = \{\langle (x,y), max(R(x,y), P(x,y)) \rangle \mid (x,y) \in X \times Y\}$.
The complement R^c of R is the fuzzy relation defined as $R^c = \{\langle (x,y), 1 - R(x,y) \rangle \mid (x,y) \in X \times Y\}$.

Exemple 2.3. *Let R and S be two fuzzy relations on $X \times X$ such that $X = \{x, y, z\}$,*
represented by the following tables

R	x	y	z
x	1.0	0.9	0.8
y	0.9	1.0	0.8
z	0.8	0.8	1.0

S	x	y	z
x	0.6	0.2	0.7
y	0.9	0.0	1.0
z	0.1	0.7	0.6

The union and intersection relations defined by

$R \cup S$	x	y	z
x	1.0	0.9	0.8
y	0.9	1.0	1.0
z	0.8	0.8	1.0

$R \cap S$	x	y	z
x	0.6	0.2	0.7
y	0.9	0.0	0.8
z	0.1	0.7	0.6

The transpose relation is given by the following table

S^t	x	y	z
x	0.6	0.9	0.1
y	0.2	0.0	0.7
z	0.7	1.0	0.6

The complementary relation is given by the following table

S^c	x	y	z
x	0.4	0.8	0.3
y	0.1	1.0	0.0
z	0.9	0.3	0.4

Définition 2.5. [1, 2, 18] *Let R, P and Q be three fuzzy relations from a universe X to a universe Y.*

(i) *if* $R \subseteq P$, *then* $R^t \subseteq P^t$;

(ii) $(R \cup P)^t = R^t \cup P^t$;

(iii) $(R \cap P)^t = R^t \cap P^t$;

(iv) $(R^t)^t = R$;

(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ *and* $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(vi) $R \cup P \supseteq R$, $R \cup P \supseteq P$, $R \cap P \subseteq R$, $R \cap P \subseteq P$;

(vii) *if* $R \supseteq P$ *and* $R \supseteq Q$, *then,* $R \supseteq P \cup Q$;

(viii) *if* $R \subseteq P$ *and* $R \subseteq Q$, *then,* $R \subseteq P \cap Q$.

Définition 2.6. [1, 2, 18] *Let R be a fuzzy relation on X. The following properties are crucial :*

(i) Reflexive : if $R(x,x)=1$, for any $x \in X$;

(ii) Irreflexive : if $R(x,x)=0$, for any $x \in X$;

(iii) Symmetrical : if $R(x,y)=R(y,x)$, for all $x,y \in X$;

(iv) Asymmetrical : if $R(x,y) \wedge R(y,x)=0$, with $x \neq y$, for all $x,y \in X$;

(v) Antisymmetry : if $(R(x,y) > 0) \wedge (R(y,x) > 0)$ then, $x = y$, for all $x,y \in X$;

(vi) Transitive : if $R(x,z) \geq max_{y\in X}(min(R(x,y),R(y,z)))$, for all $x,y,z \in X$.

Définition 2.7. ***(Composition of fuzzy relations)*** *Two fuzzy relations R and S are defined on the sets A, B, and C with $R \subseteq A \times B$ and $S \subseteq B \times C$. The composition $S \circ R$ of two relations R and S are expressed by the relation from A to C, and this composition is defined by :*

$$S \circ R(x,z) = Max_{y\in B}[Min(R(x,y),S(y,z))].$$

This composition is called Max-Min composition.

Remark 2.1. *The above definition corresponds to the one traditionally used, but it is possible to replace min with a t-norm and the max by a corresponding t-conorm.*

Exemple 2.4. *Consider the fuzzy relations $R \subseteq A \times B$ and $S \subseteq B \times C$ defined by the following tables :*

$R(.,.)$	a	b	c	d
1	*0.1*	*0.2*	*0*	*1*
2	*0.3*	*0.3*	*0*	*0.2*
3	*0.8*	*0.9*	*1*	*0.4*

$S(.,.)$	α	β	γ
a	*0.9*	*0*	*0.3*
b	*0.2*	*1*	*0.8*
c	*0.8*	*0*	*0.7*
d	*0.4*	*0.2*	*0.3*

Then the composition of R and S is defined by the following table :

$S \circ R(.,.)$	α	β	γ
1	*0.4*	*0.2*	*0.3*
2	*0.3*	*0.3*	*0.3*
3	*0.8*	*0.9*	*0.8*

2.4. Characteristics of fuzzy relations

In this section, we present some characteristics of fuzzy relations.

Définition 2.8. ***(The support of a fuzzy relations)*** *Let R be a fuzzy relation on a set X. The support of R is the crisp relation on X given by*

$$Supp(R) = \{(x, y) \in X^2 \mid R(x, y) > 0\}.$$

Définition 2.9. ***(The kernel of a fuzzy relations)*** *Let R be a fuzzy relation on a set X. The kernel of R is the crisp relation on X given by*

$$Ker(R) = \{(x, y) \in X^2 \mid R(x, y) = 1\}.$$

Définition 2.10. ***(The α-cut of a fuzzy relation)*** *Let R be a fuzzy relation on a set X. The α-cut of R is the crisp relation*

$$R_\alpha = \{(x, y) \in X^2 \mid R(x, y) \geq \alpha\} \text{ where } \alpha \in [0, 1].$$

2.5. Particular classes of fuzzy relations

2.5.1. Fuzzy equivalence relations

Définition 2.11. *A fuzzy equivalence relation R on X is a fuzzy relation that is : fuzzy reflexive, fuzzy symmetric and fuzzy transitive.*

Exemple 2.5. *Let $X = \{a, b, c, d\}$. Then the following table represents a fuzzy equivalence relation R defined on X.*

$R(.,.)$	a	b	c	d
a	1	0.8	0.7	1
b	0.8	1	0.7	0.8
c	0.7	0.7	1	0.7
d	1	0.8	0.7	1

2.5.2. Fuzzy order relations

Définition 2.12. [2, 1] *Let X be a nonempty crisp set and R be a fuzzy relation on X. R is called a fuzzy order or a partial fuzzy order if the following condition are satisfies :*

(i) *Reflexive, i.e., $R(x, x) = 1$, for any $x \in X$;*

(ii) *Antisymmetry :, i.e.,* $\begin{cases} R(x, y) > 0 \\ R(y, x) > 0 \end{cases} \Rightarrow x = y;$

(iii) *Transitivity, i.e., $R(x, z) \geq max_{y \in X}(min(R(x, y), R(y, z)))$, for all $x, y, z \in X$.*

A nonempty set X with a fuzzy order R defined on it is called a fuzzy ordered set and is denoted by $(X; R)$.

Exemple 2.6. *Let $X = \{a, b, c, d, e\}$. Then, the fuzzy relation R defined on X by $R = \{\langle (x, y), R(x, y) \rangle \mid x, y \in X\}$.*

where R is given by the following tables :

$R(.,.)$	a	b	c	d	e
a	1	0	0	0.65	0.50
b	0	1	0	0.35	0.45
c	0	0	1	0	0.80
d	0	0	0	1	0
e	0	0	0	0	1

is a fuzzy order on X.

Exemple 2.7. *Let $m, n \in \mathbb{N}$. Then, the following fuzzy relation R on $\mathbb{N}$ is an fuzzy order, where*

$$R(m,n) = \begin{cases} 1, \quad if \quad m = n, \\ 1 - \frac{m}{n}, \quad if \quad m < n, \\ 0, \quad if \quad m > n. \end{cases}$$

On the basis of the above definition of antisymmetry we define a complete (or total) fuzzy order as follows.

Définition 2.13. [18] *A fuzzy order R on a universe X is called complete (or total) if for all $x, y \in X$ it holds that $R(x;y) > 0$ or $R(y;x) > 0$.*

Exemple 2.8. *Let R be a fuzzy relation on $X = \{x, y, z\}$ given by*

R	x	y	z
x	1	0.6	0.6
y	0	1	0.4
z	0	0	1

R is a total fuzzy order.

3 Exercices

Exercice n°1 :

1. Give a technique to obtain a fuzzy version for a given mathematical notion.
2. What is the difference between crisp transitive relation and fuzzy transitive relation ?
3. Give the definition of the $t-$norm.
4. Prove that $f(x, y) = max(x, y)$ is a $t-$conorm, such that $x, y \in [0, 1]$.

Exercice n°2 :

Let R be a fuzzy relation on $X = \{x, y, z\}$ defined by :

μ_R	a	b	c
a	1	0	0
b	0.9	1	0
c	0.5	0.1	1

1. Is R a fuzzy order relation ?
2. Find : $R_{0.3}$, $Supp(R)$, $Ker(R)$.
3. Determine $S = R \circ R$.

Exercice n°3 :

Let A and B be a fuzzy subsets on $\mathbb{R}$ defined by :

$$\mu_A(x) = \frac{1}{(x-1)^2+1}.$$

$$\mu_A(x) = \frac{1}{x^2+1}.$$

1. Study the sign of $d(x) = \mu_A(x) - \mu_B(x)$.
2. Determine $A \cup B$, $A \cap B$, $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$.

Exercise 4:

Consider the fuzzy sets on the set of real numbers ℝ defined by:

$$A(x)=\begin{cases}0\ , & x<2\\ (x-2)/2\ , & 2\le x<4\\ (x-4)/2\ , & 4\le x\le 6\\ 1\ , & x>6\end{cases} \quad \text{and} \quad B(x)=\begin{cases}0\ , & x<2\\ (x-2)\ , & 2\le x<3\\ 1\ , & 3\le x<4\\ (6-x)/2\ , & 4\le x\le 6\\ 0\ , & x>6\end{cases}$$

1- Graph A and B.

2- Find $A\cap B$, $A\cup B$, A^c, B^c and graph them.

Exercise 5:

1- Prove that the binary relation given by the following table is a fuzzy order relation.

R	***a***	***b***	***c***	***d***
a	1	0.6	0.8	0.8
b	0	1	0	0.2
c	0	0.6	1	0.4
d	0	0	0	1

2- Find the following $\alpha-$cuts: R_1, $R_{0.8}$, $R_{0.6}$ *and SuppR.*

Exercise 6:

Prove that any fuzzy set A on a universe X can be written as the union of its $\alpha-$cuts, i.e.,

$$A(x)=\left(\bigcup_{\alpha\in[0,1]}\alpha\, A_\alpha\right)(x)=\sup_{\alpha\in[0,1]}\alpha.\chi_{A\alpha}(x),\quad \text{for any } x\in X.$$

Notation : $A_\alpha=\{x\in X\ \ /\ \ A(x)\ge\alpha\}$.

Exercise 7:

Let T_D be a t-norm such that

$$T_D(x,y)=\begin{cases}x, & \text{if } y=1\\ y, & \text{if } x=1\\ 0, & \text{if } x,y<1\end{cases}$$

Prove that for all t-norm T: $T_D(x,y)\le T(x,y)$, for any $x,y\in[0,1]$.

Exercice 8:

Soit A un ensemble flou dans l'ensemble des nombres réels R définit par:

$$\mu_A(x) = \begin{cases} 0 , & x \leq 2 \\ (x-2)/2 , & 2 \leq x \leq 4 \\ 1 , & x \geq 4 \end{cases}$$

1- Représenter dans un plan cet ensemble.
2- Quel est son noyau, son support et sa hauteur.

Exercice 9:

Soit R la relation floue définit sur l'ensemble des nombres réels par:

$$R(x, y) = \exp(-k(x-y)^2) \text{ tel que } K > 0 .$$

1- Montrer que R est reflexive, symétrique et non transitive.
2- Quel est son noyau et son support.

Exercice 10:

Soit X un ensemble non vide et A, B deux ensembles flous de X. Montrer que les $\alpha - coupes$ des ensembles flous ont les propriétés suivants:

1- $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$;
2- $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

Indication : Les A_α tel que $\alpha \in [0 , 1]$ sont les $\alpha - coupes$ de l'ensemble flou A, i.e., $A_\alpha = \{x \in X \ / \ \mu_A(x) \geq \alpha\}$.

Exercice 11:

Soit X un ensemble non vide et R une relation d'ordre floue sur X .

1- Montrer que la relation classique $\leq_R$ sur X définit par: $x \leq_R y \Leftrightarrow R(x, y) > 0$ est une relation d'order classique.
2- Est ce que l'ordre $\leq_R$ est total?

Bibliographie

[1] P. Burillo, H. Bustince, The fuzzy relations (Part I), Mathaware and Soft Computing Magasine, 2, (1995), 5-38.

[2] P. Burillo, H. Bustince, The fuzzy relations (Part II), Effect of Atanassov's operators on the properties of the fuzzy relations, Mathaware and Soft Computing Magasine, 2, (1995), 117-148.

[3] H. Bouremel, Clonal sets of a binary relation : Theory and Applications, Doctorat Thesis, Mohamed Boudiaf University of Msila, Algeria, (2017).

[4] J. A. Goguen, L-Fuzzy Sets, Journal of Mathematical Analysis and Applications, 18, (1967), 145–174.

[5] G. Grätzer, General Lattice Theory, Academic Press, New York, 1978.

[6] K. Menger, Statistical metrics, Proceedings of the National Academy of Sciences, USA, (1942).

[7] B.B. Meunier, La logique floue et ses application, Addison Wesley, Paris, (1995).

[8] B.B. Meunier, La logique floue, PUF collection "Que sais-je ?", (1993).

[9] S. Milles, L. Zedam, E. Rak, Characterizations of intuitionistic fuzzy ideals and filters based on lattice operations, Journal of Fuzzy Set Valued Analysis, 3 ,(2017) 143-159.

[10] S. Milles, On the intuiionistic fuzzy ordered sets, Doctoral thesis, Department of Mathematics, Mohamed Boudiaf University of Msila, Algeria, (2017).

[11] S. Milles, Étude de quelques proprits d'ordres flous intuitionistes, Mmoire de Magistre, Mohamed Boudiaf University of Msila, Algeria, (2010).

[12] A. Moufida, On fuzzy metric spaces, Memoire of Master, Mohamed Boudiaf University of Msila, Algeria, (2021).

[13] W. Näther, Copulas and t-norms, Mathematical tools for combinig probabilistic and fuzzy information, With application to error propagation and interaction, Structural Safety, 32, (2010), 366-371.

[14] B.S. Schröder, Ordered Sets, Birkhauser, Boston, 2002.

[15] W. Pedrycs, F. Gomide, An introduction to fuzzy set, Analysis and Deeign, A Bradford Book, Cambbridge, London, England, (1998).

[16] S. Willard, General Topology, Addison-Wesley Publishing Company, Massachusetts, 1970.

[17] L.A. Zadeh, Fuzzy sets, Information and Control, 8, (1965), 331-352.

[18] L.A. Zadeh, Similarity relation and fuzzy ordering, Information and orderings, Information Sciences, 3, (1971), 177-200.