الجمهوريـــــة الجزائريـــــة الديمقراطيـــة الشعبيـــة

People's Democratic Republic of Algeria

وزارة التعليـــم العالي والبحث العلمــــي

Ministry of Higher [illegible] and Scientific Research

University Center of Barika
Institute of Sciences
DEPARTMENT OF MATHEMATICS

المـركـز الجامعـي – بريكـة
مـعـهـد الـعـلـوم
قسم الرياضيات

EDUCATIONAL COURSE BOOKLET

Stability of Solutions of Ordinary Differential Equations

Prepared by:

Dr : SACI ATEF

Course / Directed Work / Previous Exams

For students of:

First-Year Master's in Discrete Mathematics and Applications, Semester 1

Academic Year: 2024/2025

Dedication

I wholeheartedly dedicate this humble work to:

all first-year master's students in mathematics,

and to all students studying these courses. *:*

ATEF

Contents

Introduction

In the 19th century, Alexander Lyapunov introduced a revolutionary approach to study the stability of solutions of ordinary differential equations (ODEs) without requiring explicit solutions. His method, based on positive definite functions, remains a fundamental tool in stability theory.

This document presents a module on **stability of ODE solutions**, intended for first-year master's students in mathematics. It is organized as follows:

- **Chapter 1**: Fundamentals of ODEs, including the Cauchy problem, linear systems (with variable/constant coefficients), and Floquet theory for periodic systems.
- **Chapter 2**: Notion of Lyapunov stability, with a focus on linear and nonlinear systems.
- **Chapter 3**: In-depth study of stability, particularly the stability of the zero solution for both linear and nonlinear systems.
- **Chapter 4**: In this chapter, you will find tutorial problems reinforcing concepts from earlier chapters along with actual exam questions from previous years.

The objective is to provide both theoretical foundations and practical applications.

Chapter 1

Reminders and Fundamental Concepts

Summary

This chapter introduces the fundamentals of differential equations, beginning with a general overview and the Cauchy problem (existence and uniqueness of solutions). We then study linear differential systems, particularly those with variable and constant coefficients, before concluding with Floquet theory for periodic systems. Special emphasis is placed on both theoretical aspects and practical applications.

1.1 Introduction

Differential equations play a crucial role in various fields of science and engineering, providing a mathematical framework for modeling dynamic systems. These equations relate a function to its derivatives, encapsulating the behavior of physical, biological, and economic phenomena.

Differential equations can be categorized into two main types: linear and nonlinear. Linear differential equations exhibit superposition, meaning that the sum of two solutions is also a solution. This property simplifies the analysis and solution of such equations. In contrast, nonlinear differential equations do not adhere to this principle, often leading to more complex behaviors and phenomena such as chaos and bifurcation.

The theory of existence and uniqueness is fundamental in the study of differential equations. It addresses whether a given differential equation has solutions and whether these solutions are unique. The famous Picard-Lindelöf theorem provides conditions under which a unique solution exists for an initial value problem, emphasizing the importance of continuity and Lipschitz conditions on the function defining the differential equation.

Understanding these concepts is essential for anyone working with differential equations, as they provide the foundation for both theoretical analysis and practical applications in engineering, physics, and beyond.

1.2 Overview of differential equations

1.2.1 Definitions

Definition 1.2.1 *(Ordinary differential equation) An ordinary differential equation, also denoted as ODE, of order* n *is a relationship between the real variable* t*, an unknown function* $t \mapsto x(t)$ *and its derivatives* $x', x'', ..., x^{(n)}$ *at the point* t*, defined by:*

$$F\left(t, x, x', ..., x^{(n)}\right) = 0,$$

where F is not independent of its last variable $x^{(n)}$. The variable t will be considered within an interval I of $\mathbb{R}$. The solution x will generally take values in $\mathbb{R}^n, n \in \mathbb{N}^*$.

Definition 1.2.2 *(Order of the Differential Equation)The order is defined as the highest derivative present in the equation.*

Example 1.2.1 *State whether the following differential equations are linear or nonlinear, and give their order (justifying your answer).*

$$(x-t)\,dt + 2tdx = 0,\ x''' = x' - t,\ \frac{d^3x}{dt^3} + t\frac{dx}{dt} - 5x = e^t$$
$$xx'' + 4x = \sin(t),\ \frac{d^2x}{dt^2} + e^t = 1,\ x''' + x^4 = \cos(t)$$

Definition 1.2.3 *(Normal differential equation) A normal differential equation of order n is defined as any equation of the form*

$$x^{(n)} = f\left(t, x, x', ..., x^{(n-1)}\right)$$

where $x^{(n)}$ denotes the n-th derivative of x with respect to t.

Example 1.2.2 *Consider the second-order differential equation:*

$$x'' = t^2 + 3x' - 2x$$

Now, let's look at a third-order differential equation:

$$x''' = e^t + x - x' + x''$$

Definition 1.2.4 *(Autonomous differential equation) An autonomous differential equation of order n is defined as any equation of the form*

$$x^{(n)} = f\left(x, x', ..., x^{(n-1)}\right)$$

In other words, f does not explicitly depend on t.

Example 1.2.3 *Second-Order autonomous Differential Equation: $x'' = -kx$,where k is a positive constant.*

Third-Order autonomous Differential Equation: $x''' = -xx'$.

Remark 1.2.1 *Autonomous equations are very important when seeking stationary solutions as well as their stability.*

Definition 1.2.5 *(Linear ordinary differential) A linear ordinary differential equation (ODE) of order n is of the form:*

$$a_n(t)\,x^{(n)}(t) + a_{n-1}(t)\,x^{(n-1)}(t) + ... + a_1(t)\,x'(t) + a_0(t)\,x(t) = g(t),$$

where all $x^{(i)}$ are of degree 1 and all coefficients depend at most on t.

Definition 1.2.6 *(Nonlinear Differential Equation) An equation is nonlinear if it does not meet the criteria for linearity. This includes cases where x or its derivatives appear to powers greater than one, are multiplied together, or are involved in nonlinear functions (like sine, exponential, etc.).*

Definition 1.2.7 *(SOLUTION) A solution to a differential equation of order n over a certain interval I of $\mathbb{R}$ is any function x defined on this interval I, differentiable n times at every point in I, and that satisfies the differential equation on I. Generally, we denote this solution as (x, I).*

Integrating a differential equation involves determining the set of all its solutions.

Example 1.2.4 *Consider the first-order differential equation: $\frac{dx}{dt} = kx$, where k is a constant. Thus, the general solution of the differential equation is: $x(t) = Ce^{kt}$, where C represents a constant determined by initial conditions. This solution is valid for any t in the interval I where x is defined and differentiable.*

1.2.2 Maximal and global solution of a first-order differential equation

Let $f : I \times \Omega \longrightarrow \mathbb{R}^n$ where Ω is an open set in $\mathbb{R}^n$. Consider the following first-order differential equation

$$x^{'}(t) = f(t, x(t)) \qquad (E)$$

Definition 1.2.8 *We say that x is a solution of (E) if there exists a non-empty interval $J \subset I$ such that:*

1- for all $t \in J$, we have $x(t) \in \Omega$.

2- x is differentiable on J and satisfies $x^{'}(t) = f(t, x(t))$ for all $t \in J$.

Definition 1.2.9 *(EXTENSION) Let $x : J \subset I \to \mathbb{R}^n$ and $\tilde{x} : \tilde{J} \subset I \to \mathbb{R}^n$ be two solutions of the same differential equation. if $J \subset \tilde{J}$ and $x = \tilde{x}$ on J then we say that $\tilde{x}$ is an extension of x.*

Example 1.2.5 *Consider the equation $x' = 2\sqrt{|x|}$. Let x be a solution defined on $J =]-1, 1[$ by $x(t) = 0$.Show that the function $\tilde{x}$ defined by*

$$\tilde{x}(t) = \begin{cases} -(t+3)^2 & si\ t \prec -3 \\ 0 & si\ -3 \leq t \leq 2 \\ (t-2)^2 & si\ t \succ 2 \end{cases}$$

is a solution that extends x.

Definition 1.2.10 *(SOLUTION MAXIMALE) A solution $x : J \subset I \to \mathbb{R}^n$ is called a maximal solution if it does not admit any extension $\tilde{x} : \tilde{J} \subset I \to \mathbb{R}^n$ with $J \subsetneq \tilde{J}$. In other words, x is a solution defined on the largest possible interval of definition (a maximal interval). Note that $J \subsetneq \tilde{J}$ means that J is strictly included in $\tilde{J}$.*

Example 1.2.6 *The function x defined on $J = \mathbb{R}$ by $x(t) = e^{-4t}$ is a maximal solution to the equation $x'(t) = -4x(t)$ because it is defined on $\mathbb{R}$, which is a maximal interval of definition.*

Lemma 1.2.1 *Let $f : I \times \Omega \longrightarrow \mathbb{R}$, where Ω is an open subset of $\mathbb{R}$, $\alpha, \beta \in \mathbb{R}$.*

1- Let x be a solution of $(x'(t) = f(t, x(t)))$ defined on $]\alpha, +\infty[$. If the $\lim\limits_{t \longrightarrow \alpha^+} x(t)$ does not exist, then x is a maximal solution.

2- Let x be a solution of $(x'(t) = f(t, x(t)))$ defined on $]-\infty, \beta[$. If the limit $\lim\limits_{t \longrightarrow \beta^-} x(t)$ does not exist, then x is a maximal solution.

Proof.

1- By contradiction, we assume that x is not maximal, which means it admits an extension $\tilde{x} : \tilde{J} \subset I \to \mathbb{R}$ with $]\alpha, +\infty[\subsetneq \tilde{J}$. Since $\tilde{J}$ is an interval, we have $\alpha \in \tilde{J}$.

On one hand, since $\tilde{x}$ is a solution of (E), it is differentiable on $\tilde{J}$.This implies that it is continuous on $\tilde{J}$. Thus, it is continuous at $t_0 = \alpha$. Therefore, $\lim\limits_{t \longrightarrow \alpha^-} \tilde{x}(t) = \tilde{x}(\alpha) \in \mathbb{R}$.On the other hand, we have $\tilde{x} = x$ on J, so $\lim\limits_{t \longrightarrow \alpha^-} x(t) = \lim\limits_{t \longrightarrow \alpha^-} \tilde{x}(t) = \tilde{x}(\alpha) \in \mathbb{R}$.This means that $\lim\limits_{t \longrightarrow \alpha^-} x(t) \in \mathbb{R}$. This leads to a contradiction because we assumed that $\lim\limits_{t \longrightarrow \alpha^-} x(t)$ does not exist.

2- Similar to (1).

Example 1.2.7 *Show that the function x defined on $J =]-\infty, 0[$ by $x(t) = \frac{1}{t}$ is a maximal solution to the differential equation $x'(t) = -x^2$.*

Definition 1.2.11 *(SOLUTION GLOBALE) If the solution x of (E) is defined on the entire interval I (i.e., $J = I$), then we say that x is a global solution.*

Example 1.2.8 *The zero function defined on $\mathbb{R}$ is a global solution to the equation $x' = x$ because it is a solution that is defined on the entire interval $I = \mathbb{R}$.*

Lemma 1.2.2 *The global solution is a maximal solution.*

Proof.

The global solution is defined on the entire interval, which is the largest possible domain of definition. This implies that it is a maximal solution.

Theorem 1.2.1 *(Regularity of the Solution) Let $f : I \times \Omega \longrightarrow \mathbb{R}^n$ and $k \in \mathbb{N}$. if $f \in C^k(I \times \Omega)$, then every solution of (E) is of class C^{k+1} on $J \subset I$.*

Proof.

By Induction

1- **Base Case** $(k = 0)$:

We want to show that if $f \in C(I \times \Omega)$, then $x \in C^1(J)$.

Since x is a solution to the equation (E), it satisfies $x^{'} = f(t,x)$. The function $f(t,x)$ is continuous on J because it is a composition of two continuous functions: f and x.

Since x' exists, x is continuous. Thus, both x' and x are continuous, which implies that $x \in C^1(J)$.

2- **Inductive Step:**

Assume that if $f \in C^k(I \times \Omega)$, then $x \in C^{k+1}(J)$.

Since x is a solution to the equation (E), it satisfies $x^{'} = f(t,x)$ which means x' is differentiable. By the inductive hypothesis, x is of class C^k with x' being continuous (as it is a composition of two continuous functions of class C^k).Therefore, x' is of class C^k, and consequently, $x \in C^{k+1}(J)$.

1.3 Existence and Uniqueness

1.3.1 Cauchy problem

Let $f : I \times \Omega \longrightarrow \mathbb{R}^n$ where Ω is an open set in $\mathbb{R}^n$. Consider the following first-order differential equation

$$x^{'}(t) = f(t, x(t)) \qquad (E)$$

Definition 1.3.1 *Any first-order ordinary differential equation (ODE) (E) equipped with an initial condition in the form:*

$$\begin{cases} x^{'}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \qquad (CP)$$

where $(t_0, x_0) \in I \times \Omega$, *is called a Cauchy problem.*

Definition 1.3.2 *A function* $x : J \subset I \rightarrow \mathbb{R}^n$ *is a solution to the initial value problem* (CP) *if it is a solution to the differential equation* $x^{'}(t) = f(t, x(t))$ *and satisfies the initial condition* $x(t_0) = x_0$.

Definition 1.3.3 *A solution* $x : J \subset I \rightarrow \mathbb{R}^n$ *is called a maximal solution of the initial value problem* (CP) *if it is a maximal solution of the differential equation* (E) *that satisfies the initial condition* $x(t_0) = x_0$.

Definition 1.3.4 *A solution* $x : J \subset I \rightarrow \mathbb{R}^n$ *is called a global solution of the initial value problem* (CP) *if it is a global solution of the differential equation* (E) *that satisfies the initial condition* $x(t_0) = x_0$.

Definition 1.3.5 *(Orbit or integral curve) The orbit (or integral curve) of the solution* Φ *to* $x^{'}(t) = f(t, x(t))$ *is the set of points:*

$$\Delta = \{(t, \Phi(t)) \ / t \in I_0 \subset I, \ \Phi(t) \in \Omega\}$$

The space $\mathbb{R}^n$ *where the solutions take their values is called the phase space.*

Remark 1.3.1 *Let* Ω *be a non-empty connected open subset of a Banach space* X *over the field of real numbers* $\mathbb{R}$.

1- Solving the Cauchy problem (CP) *locally amounts to finding an interval containing* t_0 *and a function* x *of class* C^1 *on* J *that satisfies* (CP).

2- When $f : I \times \Omega \longrightarrow X$ *is only continuous and the space* X *is infinite-dimensional, nothing can be said about the solvability of the Cauchy problem* (CP).

3- When f *is continuous and* X *is finite-dimensional, the Arzela-Peano theorem guarantees that the Cauchy system* (CP) *admits at least one solution for every* $(t_0, x_0) \in I \times \Omega$, *but uniqueness is not guaranteed in general.*

1.3.2 Questions

There are very few differential equations for which explicit solutions are known. For this reason, in this course, we will focus on the existence, uniqueness, and dependence of solutions on the initial conditions:

Under what conditions is a problem of ordinary differential equations well-posed? That is:

a) Existence: Does the equation admit a solution?

b) Uniqueness: Is this solution unique?

c) Continuous dependence on initial conditions: Does the solution depend continuously on the initial data?

These questions are fundamental to understanding whether a differential equation problem is mathematically meaningful and physically realistic.

In this section, we aim to answer these questions.

Definition 1.3.6 *(Lipschitz function) Let $C = C_1 \times C_2 \subset I \times \Omega$. We say that $f : I \times \Omega \longrightarrow \mathbb{R}^n$ is a Lipschitz function with respect to x, uniformly with respect to t, on C if there exists a constant $k > 0$ such that:*

$$\forall t \in C_1, \forall x_1, x_2 \in C_2 : \|f(t, x_1) - f(t, x_1)\| \leq k \|x_1 - x_2\|$$

Definition 1.3.7 *(Globally Lipschitz function) In the case where $C = C_1 \times C_2 = I \times \Omega$, we say that $f : I \times \Omega \longrightarrow \mathbb{R}^n$ is a globally Lipschitz function with respect to x, uniformly with respect to t.*

Definition 1.3.8 *(Locally Lipschitz Function) We say that the function $f : I \times \Omega \longrightarrow \mathbb{R}^n$ is locally Lipschitz with respect to the second variable x if: For every $(t_0, x_0) \in I \times \Omega$, there exists a neighborhood V of (t_0, x_0) and a constant $k > 0$ such that:*

$$\|f(t, x_1) - f(t, x_1)\| \leq k \|x_1 - x_2\|, \quad \forall (t, x_i) \in V, i = 1, 2 \tag{1.3.1}$$

Theorem 1.3.1 *(Mean Value Theorem) Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^m$ be a function of class C^1 (i.e., f is differentiable and its differential is continuous). Let $a, b \in U$ be two points such that the line segment $[a, b]$ is entirely contained in U, that is:*

$$[a, b] = \{a + t(a - b) \,/\, t \in [0, 1]\} \subseteq U$$

Then, there exists a point $c \in [a, b]$ (i.e., $c = a + \theta(b - a)$ for some $\theta \in (0, 1)$) such that:

$$f(a) - f(b) = Df(c) .(a - b)$$

where $Df(c)$ is the Jacobian matrix of f at the point c, and . denotes matrix multiplication.

Proposition 1.3.1 *Suppose the function $f : I \times \Omega \to \mathbb{R}^n$ is of class C^1 in a neighborhood of (t_0, x_0), where Ω is an open set in $\mathbb{R}^n$. Then f is locally Lipschitz with respect to x at (t_0, x_0). (The partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ exist and are continuous in a neighborhood of (t_0, x_0).)*

Proof.

Since f is of class C^1, the partial derivative $\frac{\partial f}{\partial x}$ is continuous in a neighborhood of (t_0, x_0). By the Mean Value Theorem applied to f as a function of x, for x_1, x_2 in a neighborhood of x_0, there exists $\theta \in [0, 1]$ such that:

$$f(t, x_1) - f(t, x_1) = \frac{\partial f}{\partial x}(t, x_2 + \theta(x_1 - x_2)) . (x_1 - x_2)$$

Taking the norm of both sides, we obtain:

$$\|f(t, x_1) - f(t, x_1)\| \leq \left\| \frac{\partial f}{\partial x}(t, x_2 + \theta(x_1 - x_2)) \right\| . \|x_1 - x_2\|$$

The continuity of $\frac{\partial f}{\partial x}$ in a neighborhood of (t_0, x_0) implies that $\|\frac{\partial f}{\partial x}\|$ is bounded in this neighborhood. Let K be an upper bound for $\|\frac{\partial f}{\partial x}\|$ in this neighborhood.

We therefore have:

$$\|f(t,x_1) - f(t,x_1)\| \leq k\,\|x_1 - x_2\|$$

which shows that f is locally Lipschitz with respect to x at (t_0, x_0).

Example 1.3.1 *Let* $y' = \frac{x}{t^2+x^2} = f(t,y)$, *with the initial condition* $x(t_0) = x_0$

where $I =]1, +\infty[$, $\Omega\,]0, +\infty[$

Show that if $f \in C^1$ *then* f *is locally Lipschitz.*

Lemma 1.3.1 *(Gronwall's Lemma / Differential Form)*

i) Let $\eta : [a,b] \to R_+$ *be a continuous function satisfying the inequality:*

$$\eta\prime(t) \leq \Phi(t).\eta(t) + \psi(t) \text{ for almost every } t \in [a,b],$$

where $\Phi, \psi : [a,b] \to R_+$ *are non-negative functions. Then, for all* $t \in [a,b]$, *the following inequality holds:*

$$\eta(t) \leq \exp\left(\int_a^t \Phi(s)ds\right)\left[\eta(a) + \int_a^t \psi(s)ds\right] \quad \forall t \in [a,b].$$

ii) In particular, if:

$$\eta\prime(t) \leq \Phi(t).\eta(t) \quad \text{on } [a,b], \text{ and } \eta(a) = 0,$$

then $\eta \equiv 0$ *on* $[a,b]$.

Proof. See Tutorial 01.

Lemma 1.3.2 *(Lemme de Gronwall / Integral Form)*

i) Let $\psi : [a,b] \to R_+$ *be a continuous function satisfying the inequality:*

$$\psi(t) \leq C_1 \int_a^t \psi(s)\,ds + C_2 \quad (p.p.t \in [a,b])$$

for C_1, $C_2 \geq 0$. *Then*

$$\psi(t) \leq C_2 e^{C_1(t-a)} \quad (p.p.t \in [a,b])$$

ii) In particular, if

$$\psi(t) \leq C_1 \int_a^t \psi(s)\, ds \quad (p.p.t \in [a,b]).$$

for $C_1 \geq 0$, *then* $\psi \equiv 0$ *on* $[a,b]$.

■

Proof. See Tutorial 01. ■

Lemma 1.3.3 *A function* $x : J \to \mathbb{R}^n$ *is a solution to the Cauchy problem (P) if and only if:*

1- x *is continuous on* J, *and for all* $t \in J$, $(t, x(t)) \in J \times \Omega \subseteq I \times \Omega$,

2- For all $t \in J$, $x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds$.

Proof. See Tutorial 01.

Theorem 1.3.2 *(Cauchy-Lipschitz Theorem, Local Existence and Uniqueness of Solutions). Let* $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, r_0,$T_0 \in \mathbb{R}_+^*$, *and*

$$f : [t_0 - T_0, t_0 + T_0] \times B_f(x_0, r_0) \to \mathbb{R}^n$$

be a continuous function. Let $k, M, T \in \mathbb{R}_+^*$. *Assume that:*

1- f *satisfies (1.3.1) for all* $t \in [t_0 - T_0, t_0 + T_0]$ *and* $x_1, x_2 \in B_f(x_0, r_0)$,

2- For all $(t, x) \in [t_0 - T_0, t_0 + T_0] \times B_f(x_0, r_0)$, $\|f(t,x)\| \leq M$,

3- $T \leq \min\left(T_0, \frac{r_0}{M}\right)$.

Then there exists a unique solution (I, x) *to the Cauchy problem*

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

where $I = [t_0 - T, t_0 + T]$. *Moreover, for any solution* (J, z) *of this Cauchy problem with* $J \subset [t_0 - T, t_0 + T]$, *we have* $z = x|_J$.

Proof.

We begin by proving the existence of a solution. If $x : [t_0 - T, t_0 + T] \to \mathbb{R}^n$ is a continuous function taking values in $B_f(x_0, r_0)$, then the function

$$t \to x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

also takes values in $B_f(x_0, r_0)$. Indeed, for $t \in [t_0 - T, t_0 + T]$, we have

$$\left\| \int_{t_0}^{t} f(s, x(s))\, ds \right\| \le \left| \int_{t_0}^{t} \| f(s, x(s)) \|\, ds \right| \le M |t - t_0| \le MT \le r_0,$$

where we have used the hypothesis that $T \le \frac{r_0}{M}$. We can therefore define a sequence $(x_m)_{m \ge 0}$ of functions from $[t_0 - T, t_0 + T]$ to $B_f(x_0, r_0)$ by setting

$$\begin{aligned} x_0(t) &= x_0, \\ x_m(t) &= x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\, ds \quad \text{for } m \ge 1. \end{aligned}$$

We will now prove, by induction on $m \ge 1$, that for all $t \in [t_0 - T, t_0 + T]$, we have

$$\| x_m(t) - x_{m-1}(t) \| \le M \frac{k^{m-1} |t - t_0|^m}{m!}.$$

Indeed, for $m = 1$, we have

$$\| x_1(t) - x_0(t) \| \le M |t - t_0|.$$

If we assume the inequality holds for some integer $m \geq 1$, then

$$\begin{aligned} \|x_{m+1}(t) - x_m(t)\| &= \left\| \int_{t_0}^{t} (f(s, x_m(s)) - f(s, x_{m-1}(s)))ds \right\| \\ &\leq \left| \int_{t_0}^{t} \|(f(s, x_m(s)) - f(s, x_{m-1}(s)))\| \, ds \right| \\ &\leq k \left| \int_{t_0}^{t} \|x_m(s) - x_{m-1}(s)\| \, ds \right| \\ &\leq k \left| \int_{t_0}^{t} M \frac{k^{m-1} |s - t_0|^m}{m!} ds \right| \\ &\leq \frac{M k^m |t - t_0|^{m+1}}{(m+1)!}, \end{aligned}$$

which proves the desired inequality for $m + 1$ and completes the induction.

From this statement, we deduce that for $t \in [t_0 - T, t_0 + T]$ and $m \geq 0$, we have

$$\|x_{m+1}(t) - x_m(t)\| \leq \frac{M k^m T^{m+1}}{(m+1)!}.$$

Therefore, the series of functions $\sum_{m \geq 0} z_m$, defined by

$$z_0 = x_0, \; z_m = x_m - x_{m-1} \text{ for } m \geq 1,$$

converges uniformly on $[t_0 - T, t_0 + T]$. This implies that the sequence of functions $(x_m)_{m \geq 0}$ converges uniformly on $[t_0 - T, t_0 + T]$. If we denote by $x : [t_0 - T, t_0 + T] \to B_f(x_0, r_0)$ its limit, then x is continuous (by uniform convergence). Moreover, by taking the uniform limit in the equality

$$x_m(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s)) \, ds,$$

we obtain that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds.$$

(To prove the uniform convergence of the function $s \mapsto f(s, x_{m-1}(s))$ to $s \mapsto f(s, x(s))$, we use the fact that f satisfies (1.3.1) on the considered domain.) In view of Lemma 1.3.3, this shows that $([t_0 - T, t_0 + T], x)$ is a solution to the considered Cauchy problem.

We now consider the question of uniqueness. Let $x : [t_0 - T, t_0 + T] \to B_f(x_0, r_0)$ be a function such that $([t_0 - T, t_0 + T], x)$ is a solution to the Cauchy problem, and we want to show that if (J, z) is a solution with $J \subset [t_0 - T, t_0 + T]$, then $z = x|_J$.

We observe that if (J, z) is such a solution, then for $t \in J$, we have

$$\begin{aligned}
\|x(t) - z(t)\| &= \left\| \int_{t_0}^{t} (f(s, x(s)) - f(s, z(s)))ds \right\| \\
&\leq \left| \int_{t_0}^{t} \|(f(s, x(s)) - f(s, z(s)))\| \, ds \right| \\
&\leq k \left| \int_{t_0}^{t} \|(x(s) - z(s))\| \, ds \right|.
\end{aligned}$$

By applying Lemma 1.3.2 to the function $s \mapsto \| x(s) - z(s) \|$ (with $A = 0$), we deduce that for all $s \in J$, we have $\| x(s) - z(s) \| = 0$, that is, $x(s) = z(s)$.

Lemma 1.3.4 *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, and let $f : \Omega \to \mathbb{R}^n$ be a continuous function that is locally Lipschitz with respect to the second variable. Let (I_1, x_1) and (I_2, x_2) be two solutions of the differential equation*

$$x'(t) = f(t, x(t))$$

If there exists $t_0 \in I_1 \cap I_2$ such that $x_1(t_0) = x_2(t_0)$, then

$$x_1|_{I_1 \cap I_2} = x_2|_{I_1 \cap I_2}.$$

Proof. See ([9], Chap. X).

Corollary 1.3.1 *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, and let $f : \Omega \to \mathbb{R}^n$ be a continuous function that is locally Lipschitz with respect to the second variable. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that $(t_0, x_0) \in \Omega$. The Cauchy problem*

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

admits a unique maximal solution. Moreover, its interval of definition is open, and any solution to this Cauchy problem can be obtained by restriction of the maximal solution.

Proof. See ([9], Chap. X).

Exercise 1.3.1 *Let* $x_0 > 0$*. Consider the following Cauchy problem:*

$$\begin{cases} x^{'} = x^2 \\ x(0) = x_0 \end{cases}$$

1- Find the explicit solution of the Cauchy problem.

2- Determine the maximal interval of definition of this solution.

3- Show that the maximal solution is not global, i.e., it is not defined on all of $\mathbb{R}$*.*

We present here the Cauchy-Peano-Arzelà Theorem, a key result in the theory of ordinary differential equations. This theorem guarantees the existence of at least one solution to the Cauchy problem, though it does not ensure uniqueness.

Theorem 1.3.3 *(Cauchy-Peano-Arzelà Theorem) Let* $t_0 \in \mathbb{R}$*,* $x_0 \in \mathbb{R}^n$*,* r_0*,*$T_0 \in \mathbb{R}_+^*$*, Assume that the function* $f : [t_0, t_0 + T_0] \times B_f(x_0, r_0) \to \mathbb{R}^n$ *is continuous, where* $B_f(x_0, r_0)$ *is the closed ball in* $\mathbb{R}^n$ *centered at* x_0 *with radius* r_0*.*

Furthermore, suppose there exists a constant $M > 0$ *such that:*

$$\|f(t,x)\| \leq M \quad \text{For all } (t,x) \in [t_0, t_0 + T_0] \times B_f(x_0, r_0).$$

Then, the Cauchy problem:

$$\begin{cases} x^{'}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

has a solution $x(t)$ *defined on the interval* $[t_0, t_0 + T_0]$*, where:* $T = \min\left(T_0, \frac{r_0}{M}\right).$

Proof. See ([9], Chap. X)

Example 1.3.2 *Consider the Cauchy problem:*

$$\begin{cases} x'(t) = 3x^{\frac{2}{3}} \\ x(0) = 0 \end{cases} \qquad t \geq 0 \tag{1.3.2}$$

This problem admits two solutions on the same time interval J:

$$x_1(t) = 0, \;\; J = \mathbb{R},$$

$$x_2(t) = t^3, \;\; J = \mathbb{R}.$$

Thus, the problem (1.3.2) does not have a unique solution on $\mathbb{R}$.

Theorem 1.3.4 *(Global Existence and Uniqueness) Let $f : \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, be locally Lipschitz. Then, for every $(t_0, x_0) \in \Omega$, the differential equation $x'(t) = f(t, x(t))$ has one and only one maximal solution $\varphi : I \subset \mathbb{R} \to \mathbb{R}^n$ satisfying the initial condition (t_0, x_0). The interval I is open.*

Proof. See [10].

The following result provides information about the behavior of maximal solutions.

Theorem 1.3.5 *Let $f : \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ is open, be locally Lipschitz in x, and let $\varphi :]a, b[\to \mathbb{R}^n$ be a maximal solution of the equation $x'(t) = f(t, x(t))$. Then, for every compact set $K \subset \Omega$, there exist a_K, b_K with $a < a_K < b_K < b$ such that $(t, \varphi(t)) \notin K$ for all tt satisfying $a < t < a_K$ or $b_K < t < b$.*

What this theorem tells us, in particular, is that if $\Omega = \mathbb{R} \times \mathbb{R}^n$ and $b < \infty$, then as $t \to b$, $(t, \varphi(t))$ must leave every compact subset of $\mathbb{R} \times \mathbb{R}^n$, and therefore necessarily $\|\varphi(t)\| \to \infty$.

Proof. See [10].

Theorem 1.3.6 . *(Complement to the Global Existence and Uniqueness Theorem) Let $\Omega =]t_1, t_2[\times \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be a continuous function such that for every τ_1,τ_2 with $t_1 < \tau_1 <$*

$\tau_2 < t_2$, *there exists a constant* K_{τ_1,τ_2} *satisfying*

$$\| f(t, x_1) - f(t, x_1) \| \leq K_{\tau_1,\tau_2} \| x_1 - x_2 \|, \quad \forall (t, x_i) \in [\tau_1, \tau_2] \times \mathbb{R}^n, i = 1, 2$$

Then, for every $(t_0, x_0) \in]t_1, t_2[\times \mathbb{R}^n$, *there exists one and only one solution* $\phi :]t_1, t_2[\to \mathbb{R}^n$ *satisfying the initial condition* (t_0, x_0).

Proof. See [10].

The following theorem is an easy consequence of Theorem 1.3.5.

Theorem 1.3.7 *(Continuous Dependence on Initial Conditions and Parameters). Let* $f : \Omega \to \mathbb{R}^n$, *where* $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ *is an open set, and* f *is locally Lipschitz in* x. *Let* $(t_0, x_0) \in \Omega$, *and let* $\varphi_{(t_0,x_0)} : I_{(t_0,x_0)} \to \mathbb{R}^n$ *be the maximal solution with initial condition* (t_0, x_0). *Then, for every closed and bounded interval* $I \subset I_{(t_0,x_0)}$ *with* $t_0 \in I$, *and for every* $\epsilon > 0$, *there exist* $\delta_1^\epsilon, \delta_2^\epsilon > 0$ *such that if*

$$|t_1 - t_0| < \delta_1^\epsilon \quad \text{and} \quad \|x_1 - x_0\| < \delta_2^\epsilon,$$

and if $\varphi_1 : I_1 \to \mathbb{R}^n$ *denotes the maximal solution with initial condition* (t_1, x_1), *then:*

$$I \subset I_1,$$
$$\|\varphi_1(t) - \varphi(t)\| < \epsilon \quad \text{for all } t \in I.$$

Proof. See [10].

Let us define the flow Φ of the equation by:

$$U = \left\{ (t_0, x_0, t) \in \Omega \times \mathbb{R} / \ t \in I_{(t_0,x_0)} \right\}$$

and

$$\Phi : U \to \mathbb{R}^n, \ \Phi(t_0, x_0, t) = \varphi_{(t_0,x_0)}(t).$$

Here, Φ represents the family of all maximal solutions. The following corollary is an immediate consequence of the previous theorem.

Corollary 1.3.2 *U is an open set, and Φ is continuous.*

1.4 Linear Differential Systems

1.4.1 Linear systems with variable coefficients

In this section, we will address systems of ODEs, which can be obtained directly by modeling a problem involving several unknown functions, or by converting an nn-th order ODE into a system of several first-order ODEs. We will deal here only with the specific case of linear systems.

Definition 1.4.1 *A first-order linear differential system in $\mathbb{R}^n$ defined on an open interval $I \subset \mathbb{R}$ is a system of the form:*

$$\begin{cases} x_1'(t) = a_{11}(t)\,x_1(t) + ... + a_{1n}(t)\,x_n(t) + b_1(t) \\ x_2'(t) = a_{21}(t)\,x_1(t) + ... + a_{2n}(t)\,x_n(t) + b_2(t) \\ \quad . \\ \quad . \\ \quad . \\ x_n'(t) = a_{n1}(t)\,x_1(t) + ... + a_{nn}(t)\,x_n(t) + b_n(t) \end{cases} \qquad (S)$$

where the functions a_{ij}: $I \to \mathbb{R}$ and b_i: $I \to \mathbb{R}$ are given. We always assume that these functions are continuous on the interval I.

For every $t \in I$, let:

$$B(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ . \\ . \\ b_n(t) \end{pmatrix}, \quad A(t) = (a_{ij}(t)) \in M_n(\mathbb{R}), \quad \text{and } X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{pmatrix}.$$

The system can then be written in matrix form as:

$$X'(t) = A(t)X(t) + B(t) \qquad (S)$$

Definition 1.4.2 *A solution to (S) is a function $X : J \to \mathbb{R}^n$, that is, a parametrized curve in $\mathbb{R}^n$, defined on an open interval $J \subset I$, differentiable at every point $t \in J$, and satisfying (S) for all $t \in J$ (thus, the function X is of class C^1 on J).*

Definition 1.4.3 *The homogeneous system associated with (S) is the linear differential system defined by:*

$$X'(t) = A(t)X(t) \qquad (SH)$$

As with first-order differential equations, we will use the notions of maximal solution and global solution for the differential system (S).

We have the following result concerning the existence and uniqueness of solutions to the Cauchy problem for (S):

Theorem 1.4.1 *(Cauchy-Lipschitz) Let I be an open interval and a_{ij}: $I \to \mathbb{R}$ and b_i: $I \to \mathbb{R}$ be continuous functions on I. Then, for all $t_0 \in I$ and $X_0 \in \mathbb{R}^n$, the Cauchy problem*

$$\begin{cases} X'(t) = A(t)X(t) + B(t) \\ X(t_0) = X_0 \end{cases}$$

has a unique maximal solution. This solution is global (defined on I).

Proof. See Tutorial 02.

Remark 1.4.1 *Every maximal solution of (S) or (SH) is global: therefore, we will only consider solutions defined on I.*

As with linear differential equations, we also have the superposition principle in the context of linear differential systems.

Proposition 1.4.1 *If $X_1 : I \to \mathbb{R}^n$ and $X : I \to \mathbb{R}^n$ are two solutions of the system (S), then $Z = X - X_1$ is a solution of the associated homogeneous system (SH). Therefore, any solution X of (S) can be expressed in the form: $X = X_1 + Z$ where Z is a solution of (SH).*

Conversely, if Z is any solution of (SH), then $X_1 + Z$ is a solution of (S).

Proof. See [2].

Therefore:

To find all the solutions of the linear differential system (S), it is sufficient to:

1- Determine all the solutions Z of the associated homogeneous system.

2- Determine any particular solution X_1 of (S).

The solutions of (S) are then all the functions of the form: $X = X_1 + Z$.

Proposition 1.4.2 *The set S_H of solutions to the homogeneous linear differential system (SH) is a vector subspace of the $\mathbb{R}$-vector space $C^1(I, \mathbb{R}^n)$.*

Proof. The set S_H is a subspace of $C^1(I, \mathbb{R}^n)$ because:

- The zero function (which is trivially a solution) belongs to (SH).

- If Z_1 and Z_2 are solutions, then $Z_1 + Z_2$ is also a solution (linearity).

- If Z is a solution and $c \in \mathbb{R}$, then cZ is also a solution (scalar multiplication).

Thus, S_H satisfies the properties of a vector subspace.

Proposition 1.4.3 *The set S_H of solutions to the homogeneous linear differential system (SH) in $\mathbb{R}^n$ is an $\mathbb{R}$-vector space of dimension n.*

Proof. Let us show that $\dim S_H = n$: Fix $t_0 \in I$, and consider the map ϕ_{t_0} defined as follows:

$$\phi_{t_0} : S_H \longrightarrow \mathbb{R}^n$$
$$Z \longrightarrow \phi_{t_0}(Z) = Z(t_0)$$

It suffices to show that ϕ_{t_0} is an isomorphism (i.e., a bijective linear map) between S_H and $\mathbb{R}^n$. This implies that $\dim S_H = \dim \mathbb{R}^n$. Since $\dim \mathbb{R}^n = n$, it follows that $\dim S_H = n$.

Lemma 1.4.1 *(Wronskian): Let $X_1, \ldots, X_n : I \to \mathbb{R}^n$ be solutions of (SH). Then the following three propositions are equivalent:*

1- $X_1, \ldots, X_n$ are linearly independent.

2-There exists a $t_0 \in I$ such that the matrix defined by

$$\left(X_1(t_0) | , ..., X_n(t_0) | \right), \tag{1.4.1}$$

is invertible.

3- The matrix

$$\left(X_1(t) | , ..., X_n(t) | \right), \tag{1.4.2}$$

is invertible for all $t \in I$.

The determinant of the matrix (1.4.2) *is called the Wronskian.*

Proof. See [4].

Definition 1.4.4 *(FUNDAMENTAL MATRIX): Let $X_1, \ldots, X_n : I \to \mathbb{R}^n$ be solutions of (SH). If the n functions are linearly independent, they are said to form a fundamental set of* *solutions of (SH). We then denote*

$$M(t) = (X_1(t)|, ..., X_n(t)|), \tag{1.4.3}$$

the $n \times n$ matrix, which we call the fundamental matrix of the system (SH).

Theorem 1.4.2 *(HOMOGENEOUS SYSTEM SOLUTIONS) Let $X_1, X_2, \ldots, X_n$ be a fundamental set of solutions to (SH). Then any solution X of (SH) is of the form*

$$X(t) = \sum_{i=1}^{n} c_i X_i(t)$$

where $c_1, c_2, ..., c_n \in \mathbb{R}$.

Proof. See [4].

Theorem 1.4.3 *(Solutions of Non-Homogeneous Systems) Let $X_1, X_2, \ldots, X_n$ be a fundamental set of solutions for the homogeneous problem (SH), and let X_p be a particular solution of (S). Then, any solution X of (S) can be expressed in the form:*

$$X(t) = X_p + \sum_{i=1}^{n} c_i X_i(t)$$

where $c_1, c_2, ..., c_n \in \mathbb{R}$.

Proof. See [4].

Theorem 1.4.4 *Let M be a fundamental matrix of the system (SH). Then:*

1- For all $t \in \mathbb{R}$, we have $M'(t) = A(t)M(t)$.

2- The general solution of (SH) is given by $X = MC$, where $C \in \mathbb{R}^n$.

Proof. Let M be a fundamental matrix of the system (SH). Then, $M = (X_1, X_2, \ldots, X_n)$, where $\{X_1, X_2, \ldots, X_n\}$ is a fundamental system of solutions of (SH).

1- We have:

$$M'(t) = (X_1(t), X_2(t), \ldots, X_n(t))'$$

This means:

$$M'(t) = (X_1'(t), X_2'(t), \ldots, X_n'(t))$$

Since each $X_i(t)$ is a solution of (SH), we know that $X_i'(t) = A(t)X_i(t)$. Therefore:

$$M'(t) = (A(t)X_1(t), A(t)X_2(t), \ldots, A(t)X_n(t))$$

This can be rewritten as:

$$M'(t) = A(t)(X_1(t), X_2(t), \ldots, X_n(t)) = A(t)M(t)$$

Thus, we have shown that:

$$M'(t) = A(t)M(t)$$

2- The general solution $X(t)$ of the system(SH) can be expressed as a linear combination of the fundamental solutions $X_1(t), X_2(t), \ldots, X_n(t)$. That is:

$$X(t) = c_1X_1(t) + c_2X_2(t) + ... + c_nX_n(t)$$

This can be written in matrix form as:

$$X(t) = (X_1(t), X_2(t), \ldots, X_n(t))C$$

where C is the column vector of constants.

Thus, the general solution is:

$$M'(t) = M(t)C$$

Theorem 1.4.5 *(Solution to the Cauchy Problem): Let $M(t)$ be a fundamental matrix of solutions to the homogeneous system (SH), and let $B(t)$ be a continuous vector function. If the* *initial condition $X(t_0) = X_0$ is specified at $t_0 \in I$ and $X_0 \in \mathbb{R}^n$, then the unique solution to the Cauchy problem:*

$$\begin{cases} X'(t) = A(t)\,X(t) + B(t), \\ X(t_0) = X_0. \end{cases} \tag{1.4.4}$$

is given by:

$$X(t) = M(t)M^{-1}(t_0)\,X_0 + M(t)\int_{t_0}^{t} M^{-1}(s)B(s)\,ds \tag{1.4.5}$$

Proof. by theorem 1.4.3 we have

$$X(t) = X_p + \sum_{i=1}^{n} c_i X_i(t)$$

To find a particular solution X_p to the nonhomogeneous system, we use the method of variation of constants. Here are the detailed steps:

Step 1: We seek a particular solution X_p in the form:

$$X_p = \sum_{i=1}^{n} X_i(t)\,\gamma_i(t)$$

where:

$X_i(t)$ are the columns of the fundamental matrix $M(t)$ (solutions to the homogeneous system),

γ_i: $I \to \mathbb{R}$ are functions to be determined.

Step 2: By differentiating X_p, we obtain:

$$X_p'(t) = \sum_{i=1}^{n} X_i'(t)\,\gamma_i(t) + \sum_{i=1}^{n} X_i(t)\,\gamma_i'(t)$$

Since $X_i(t)$ are solutions to the homogeneous system $X'(t) = A(t)\,X(t)$, we have $X_i'(t) = A(t)\,X_i(t)$. Thus:

$$X_p'(t) = A(t)\sum_{i=1}^{n} X_i(t)\,\gamma_i(t) + \sum_{i=1}^{n} X_i(t)\,\gamma_i'(t).$$

This can be rewritten as:

$$X_p'(t) = A(t)\,X_p(t) + M(t)\,\gamma'(t),$$

where $M(t)$ is the fundamental matrix and $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))^T$.

Step 3: We want X_p to be a solution to the nonhomogeneous system:

$$X_p'(t) = A(t) X_p(t) + B(t),$$

Comparing this with the expression obtained in Step 2, we deduce that:

$$M(t)\gamma'(t) = B(t).$$

Step 4: Since $M(t)$ is invertible (as it is a fundamental matrix), we can write:

$$\gamma'(t) = M^{-1}(t) B(t),$$

Integrating this equation, we obtain:

$$\gamma(t) = \int_{t_0}^{t} M^{-1}(s) B(s)\, ds.$$

where $t_0 \in I$ is fixed.

Step 5: Substituting $\gamma(t)$ into the expression for X_p, we find:

$$X_p(t) = M(t)\gamma(t) = M(t)\int_{t_0}^{t} M^{-1}(s) B(s)\, ds.$$

Step 6: According to Theorem 1.4.3, the general solution to the nonhomogeneous system is the sum of a particular solution X_p and the general solution to the homogeneous system X_H. Thus:

$$X(t) = X_p(t) + X_H = M(t)\int_{t_0}^{t} M^{-1}(s) B(s)\, ds + M(t) C$$

where $C \in \mathbb{R}^n$ is a constant vector.

Step 7: (Solution to the Cauchy Problem)

If we fix an initial condition $X(t_0) = X_0$, then the vector C is determined by:

$$X(t_0) = M(t_0) C = X_0$$

Since $M(t_0)$ is invertible, we have:

$$C = M^{-1}(t_0) X_0.$$

The solution to the Cauchy problem is therefore given by:

$$X(t) = M(t)M^{-1}(t_0) X_0 + M(t) \int_{t_0}^{t} M^{-1}(s) B(s)\, ds.$$

Corollary 1.4.1 *The solution to the system* (S)

$$X'(t) = A(t) X(t) + B(t),$$

is given by:

$$X(t) = X_p(t) + X_H = M(t) \int_{t_0}^{t} M^{-1}(s) B(s)\, ds + M(t) C$$

Corollary 1.4.2 *Let* $(t_0, X_0) \in I \times \mathbb{R}^n$. *The solution to the system*

$$\begin{cases} X'(t) = A(t) X(t), \\ X(t_0) = X_0. \end{cases} \tag{1.4.6}$$

is given by:

$$X(t) = M(t)M^{-1}(t_0) X_0 \tag{1.4.7}$$

Remark 1.4.2 *The matrix* $M(t)M^{-1}(t_0)$ *is called the resolvent matrix of the system (1.4.6). It is denoted by* $R(t, t_0)$. *i.e*

$$R(t, t_0) = M(t)M^{-1}(t_0)$$

1.4.2 Linear systems with constant coefficients

In this section, we will consider a special case of the previous section. We will study problem (S) with A being a constant. The methods of resolution for such systems rely heavily on the properties of the matrix exponential and the structure of the matrix A. Specifically, we will explore how to find solutions using techniques such as diagonalization, and the matrix exponential e^{tA}. These methods are particularly effective when A is a constant matrix, as they allow us to derive explicit formulas for the solutions.

Matrix Exponential Method

The goal is to focus on finding a fundamental matrix $M(t) \in M_n(\mathbb{R})$ for (S). We will use the concept of the matrix exponential, which we will explain here.

We also recall that one definition of the exponential function e^x, where $x \in \mathbb{R}$, is:

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

We will see that this definition also applies to matrix exponentials.

Definition 1.4.5 *The matrix exponential of a square matrix $A \in M_n(\mathbb{R})$ is defined as:*

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + ... = \sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

where: I is the identity matrix of the same size as A.

Lemma 1.4.2 *The series $\sum_{n=0}^{+\infty} \frac{A^n}{n!}$ converges. This is defined using the operator norm*

$$\|A\| = \sup_{X \in \mathbb{R}^n, X \neq 0} \frac{\|AX\|}{\|X\|}$$

where $\|.\|$ represents any vector norm on $\mathbb{R}^n$.

Proof. For any k, $\left\|A^k\right\| \leq \|A\|^k$. We examine the absolute series of the original series:

$$\sum_{n=0}^{+\infty} \frac{\|A^n\|}{n!} \leq \sum_{n=0}^{+\infty} \frac{\|A\|^n}{n!}$$

The series on the right-hand side is the ordinary exponential series $e^{\|A\|}$, which is a convergent series for all finite values of $\|A\|$.

the original series $\sum_{n=0}^{+\infty} \frac{A^n}{n!}$ converges absolutely, and therefore converges in the space of matrices $M_n(\mathbb{R})$.

Remark 1.4.3 *We accept that the exponential of the zero matrix is the identity matrix, i.e.,*

$$e^{0_{M_n(\mathbb{R})}} = I,$$

where $0_{M_n(\mathbb{R})}$ *is the zero matrix and* I *is the identity matrix of the same size.*

Proposition 1.4.4 *Let* $A, B \in Mn(\mathbb{R})$.

1- If $AB = BA$, *then* $e^{A+B} = e^A e^B$.

2- For any matrix A, *we have* $\left(e^A\right)^{-1} = e^{-A}$.

3- Let $\lambda \in R$. *Then,* $e^{\lambda I_n + A} = e^{\lambda} e^A$.

Proof. See [12].

The question that arises then is: How can we find e^A?

1) If A is a diagonal matrix with a_{ii} on the diagonal, then e^A is the diagonal matrix with $e^{a_{ii}}$ on the diagonal. i.e, if

$$A = \begin{pmatrix} a_{11} & 0 & . & . & 0 \\ 0 & a_{22} & . & . & 0 \\ . & . & . & & \\ . & . & & . & \\ 0 & 0 & . & . & a_{nn} \end{pmatrix} \quad \text{then} \quad e^A = \begin{pmatrix} e^{a_{11}} & 0 & . & . & 0 \\ 0 & e^{a_{22}} & . & . & 0 \\ . & . & . & & \\ . & . & & . & \\ 0 & 0 & . & . & e^{a_{nn}} \end{pmatrix}$$

2) If $A = PDP^{-1}$, where D is a diagonal matrix, then $e^A = Pe^D P^{-1}$.

3) Let N be a nilpotent matrix of index $m \in N^*$ (meaning $N^m = 0$ for some positive integer m). Then,

$$e^N = I_n + \frac{N}{1!} + \frac{N^2}{2!} + ... + \frac{N^{m-1}}{(m-1)!}.$$

4) Suppose that $A = \lambda I + N$, where N is an upper triangular matrix with zeros on the diagonal. Then, $N^m = 0$ for all $m \geq n$. Consequently,

$$e^A = e^{\lambda} \sum_{k=0}^{m-1} \frac{N^k}{k!}$$

Theorem 1.4.6 *(Fundamental Solution) The matrix $M(t) = e^{tA} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} A^n$ is a fundamental solution of (SH). It is therefore invertible and satisfies $M'(t) = AM(t)$.*

Proof. By Lemma 1.4.2, the series $\sum_{n=0}^{+\infty} \frac{t^n}{n!} A^n$ converges for any matrix $A \in M_n(\mathbb{R})$ and any $t \in \mathbb{R}$.

Differentiating term by term, we obtain:

$$\begin{aligned} M'(t) &= \frac{d}{dt} \sum_{n=0}^{+\infty} \frac{t^n}{n!} A^n \\ &= A \sum_{n=1}^{+\infty} \frac{t^{n-1}}{(n-1)!} A^{n-1} \quad p = n-1 \\ &= A \sum_{p=0}^{+\infty} \frac{t^p}{(p)!} A^p \\ &= A\ e^{tA} = AM(t) \end{aligned}$$

The fundamental solution of a linear system is always invertible for all $t \in \mathbb{R}$. Therefore, $M(t)$ is invertible for all $t \in \mathbb{R}$.

Theorem 1.4.7 *(Solution to Homogeneous Systems): The general solution of the system $X'(t) = AX(t)$ is given by*

$$X(t) = e^{At} C,$$

where $C \in \mathbb{R}^n$.

Proof. $X(t) = e^{At}C$ is a solution due to the properties of the matrix exponential.

Theorem 1.4.8 *(Solution to Non-Homogeneous Systems): If* A *is a constant matrix, then the solution to* (1.4.4) *is given by*

$$X(t) = e^{A(t-t_0)}X_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)\,ds.$$

Proof. The proof of this theorem is the same as the proof of Theorem 1.4.5. Both rely on the variation of parameters method and the properties of the matrix exponential to derive the solution to the non-homogeneous system.

Corollary 1.4.3 *Let* $(t_0, X_0) \in I \times \mathbb{R}^n$. *The solution to the system*

$$\begin{cases} X'(t) = AX(t), \\ X(t_0) = X_0. \end{cases}$$

is given by:

$$X(t) = e^{A(t-t_0)}X_0.$$

Spectral Method

The spectral method is a powerful tool for solving systems of linear differential equations with constant coefficients. Consider the system $X'(t) = AX(t)$, where A is a square matrix with constant coefficients. The eigenvalues and eigenvectors of A play a key role in constructing solutions to this system.

Lemma 1.4.3 *Let* $A \in M_n(\mathbb{R})$, $\lambda \in \mathbb{R}$ *be an eigenvalue of* A, *and* $V \in \mathbb{R}^n$ *be an eigenvector associated with* λ. *Then, the function* $X : \mathbb{R} \to \mathbb{R}^n$ *defined by* $X(t) = e^{\lambda t}V$ *is a solution to the system* $X'(t) = AX(t)$.

Proof. By differentiating $X(t)$, we obtain $X'(t) = \lambda e^{\lambda t} V$. On the other hand, multiplying A by $X(t)$ gives $A.X(t) = A.(e^{\lambda t} V) = e^{\lambda t} A.V$. Since V is an eigenvector associated with λ, we have $AV = \lambda V$, which implies $A.X(t) = e^{\lambda t} \lambda.V$. Therefore, $X'(t) = AX(t)$ for all $t \in \mathbb{R}$, proving that $X(t)$ is indeed a solution to the system.

Theorem 1.4.9 *If A admits n linearly independent eigenvectors $V_1, V_2, \ldots, V_n$ associated with the real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the general solution of the homogeneous system (SH) is given, for all $t \in \mathbb{R}$, by:*

$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} + ... + c_n V_n e^{\lambda_n t}$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Proof. To prove the result of the theorem, we define for all $t \in \mathbb{R}$:

$$X_1(t) = V_1 e^{\lambda_1 t},\ X_2(t) = V_2 e^{\lambda_2 t}, ..., X_n(t) = V_n e^{\lambda_n t}.$$

To show that the general solution of the homogeneous system (SH) is given by the linear combination of these functions, it is sufficient to prove that $\{X_1, X_2, \ldots, X_n\}$ forms a fundamental system of solutions for (SH).

Proof that $X_1, X_2, \ldots, X_n$ are solutions of (SH):

From the previous lemma, each $X_i(t) = V_i e^{\lambda_i t}$ is a solution of (SH) because V_i is an eigenvector of A associated with the eigenvalue λ_i.

Proof that $X_1, X_2, \ldots, X_n$ are linearly independent:

Consider the Wronskian $W(t)$ of the system at $t = 0$:

$$W(0) = \det(X_1(0), X_2(0), ..., X_n(0)) = \det(V_1, V_2, ..., V_n).$$

since $V_1, V_2, \ldots, V_n$ are linearly independent (by assumption), the determinant $\det(V_1, V_2, ..., V_n)$ is non-zero:

$$W(0) \neq 0$$

this ensures that the solutions $X_1, X_2, \ldots, X_n$ are linearly independent.

ince $X_1, X_2, \ldots, X_n$ are solutions of (SH) and linearly independent, they form a fundamental system of solutions. Therefore, the general solution of (SH) is given by:

$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} + ... + c_n V_n e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Corollary 1.4.4 *If A admits n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the general solution of the homogeneous system (SH) is given, for all $t \in \mathbb{R}$, by:*

$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} + ... + c_n V_n e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Here, $V_1, V_2, \ldots, V_n$ are the eigenvectors associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

Proof. Since A admits n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the corresponding eigenvectors are linearly independent. Therefore, we conclude by applying the previous theorem.

1.5 The Floquet theory

The Floquet theorem is a fundamental result in the study of systems of linear differential equations with periodic coefficients. It is particularly useful for analyzing the stability of solutions to such systems.

Theorem 1.5.1 *(Floquet 1883) Consider a system of linear differential equations of the form:*

$$X'(t) = A(t)X(t) \tag{1.5.1}$$

where $A(t)$ is a square matrix whose elements are periodic functions with period T, that is,

$$A(t+T) = A(t) \quad \textit{for all } t \in \mathbb{R}.$$

The fundamental matrix $\Phi(t,0)$ of the system (1.5.1) can be expressed in the form:

$$\Phi(t,0) = P(t)\, e^{Bt}$$

where:

- $P(t)$ is a T-periodic matrix, meaning $P(t+T) = P(t)$, and $P(0) = I$ (the identity matrix).

- B is a constant matrix.

Proof.(main idea):

1- We start by considering the fundamental matrix $\Phi(t,0)$, which is a solution to the system $X'(t) = A(t)X(t)$ with $\Phi(0,0) = I$ (the identity matrix).

2- Using the periodicity of $A(t)$, we show that $\Phi(t+T,0)$ is also a fundamental solution.

3- Consequently, there exists a constant matrix C such that $\Phi(t+T,0) = \Phi(t,0)C$.

4- By diagonalizing C (or using its Jordan form), we can write $C = e^{BT}$, where B is a constant matrix.

5- We then define $P(t) = \Phi(t,0)e^{-Bt}$ and verify that $P(t)$ is periodic with period T and that $P(0) = I$.

Chapter 2

The Concept of Stability

Summary

This chapter focuses on the Lyapunov stability analysis of solutions to ordinary differential equations, with special emphasis on linear systems

2.1 Introduction

By definition, stability implies that if a system is in equilibrium, it will remain in this state as time evolves. Lyapunov stability analysis involves studying the system's trajectories when the initial state is close to an equilibrium point.

The key objective of stability theory is to draw conclusions about the system's behavior without explicitly computing its trajectories.

2.2 The case of ordinary differential equations

To explore classical results on Lyapunov stability, we first recall the following fundamental definitions.

We denote by

- Ω a non-empty open subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}^*$),

- I a non-empty interval of $\mathbb{R}$, unbounded on the right.

For a continuous function $f : \Omega \to \mathbb{R}^n$, we associate the autonomous system:

$$x' = f(x), \; x' = \frac{dx}{dt} \tag{2.2.1}$$

For a continuous function $f : I \times \Omega \to \mathbb{R}^n$, we associate the non-autonomous system:

$$x' = f(t, x) \tag{2.2.2}$$

The notation $x(t, t_0, x_0) = x(t)$ refers to a solution $x(t)$ of system (2.2.1) or (2.2.2) such that $x(t_0) = x_0$.

Definition 2.2.1 *For the autonomous system (2.2.1) :*

a point $x^* \in \Omega$ is called an equilibrium point (or fixed point) if:

$$f(x^*) = 0.$$

This means that if the system starts at x^*, it remains there for all time, i.e., $x(t) = x^*$ is a constant solution.

Definition 2.2.2 *For the non-autonomous system (2.2.2) :*

an equilibrium point is a point $x^* \in \Omega$ such that:

$$f(t, x^*) = 0, \qquad \forall t \in I.$$

This means $x(t) = x^*$ is a constant solution for all time t.

Example 2.2.1 *Consider the following system:*

$$\begin{cases} x_1' = x_2 \\ x_2' = -(1 - x_2^2)x_2 - x_1 - x_1^2 \end{cases}$$

Determine the equilibrium points of the system.

Finding Equilibrium Points

An equilibrium point (x_1^*, x_2^*) satisfies:

From $x_1' = 0$

$$x_2 = 0.$$

Substitute $x_2 = 0$ into $x_2' = 0$:

$$-x_1 - x_1^2 = 0 \Longrightarrow x_1 = 0, \; x_1 = -1.$$

Equilibrium Points: $(x_1^*, x_2^*) = (0, 0), (x_1^*, x_2^*) = (-1, 0),$

Definition 2.2.3 *(Stability, [7])*

We say that an equilibrium point $x = x^*$ is stable (in the sense of Lyapunov) if:

$$\forall \varepsilon > 0, \forall t_0 \geq 0, \exists \delta = \delta(t_0, \varepsilon) > 0, \textit{ such that}$$
$$\| x_0 - x^* \| < \delta \Longrightarrow \| x(t) - x^* \| < \varepsilon, \forall t \geq t_0.$$

Specifically for $n = 2$

Consider the second-order system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= f_1(t, x, y) \\ \frac{dy}{dt} &= f_2(t, x, y) \\ x(t_0) &= x_0, \;\; y(t_0) = y_0 \end{aligned} \tag{2.2.3}$$

The equilibrium point $(x = y = 0)$ is stable (in the sense of Lyapunov for t $\to +\infty$) if:

$$\forall \varepsilon > 0, \forall t_0 \geq 0, \exists \delta = \delta(t_0, \varepsilon) > 0, \textit{ such that for any solution of } (2.2.3), (x(t), y(t))$$
$$|x_0| < \delta, \textit{ and } |y_0| < \delta \Longrightarrow |x(t)| < \varepsilon \textit{ and } |y(t)| < \varepsilon, \forall t \geq t_0.$$

Example 2.2.2 *Study the stability of the solution of the following system:*

$$\begin{aligned} x'(t) &= -9y \\ y'(t) &= x \\ x(t_0) &= 0, \;\; y(t_0) = 0 \end{aligned} \tag{2.2.4}$$

such that, the general solution of the system (2.2.4) is

$$x(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t)$$
$$y(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

Definition 2.2.4 *(Asymptotic Stability, [7])*

An equilibrium point x^* is said to be asymptotically stable (AS) if:

1- It is stable (in the sense of Lyapunov).

2- It is attractive:

$$\exists \delta_a(t_0) > 0 \text{ such that } \| x_0 - x^* \| < \delta_a \Longrightarrow \lim_{t \to +\infty} x(t) = x^*.$$

Definition 2.2.5 *(Uniform Stability, [7])*

An equilibrium point $x = x^*$ is said to be uniformly stable (US) if:

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \text{ such that } \forall t_0 \geq 0,$$

$$\| x_0 - x^* \| < \delta \Longrightarrow \| x(t) - x^* \| < \varepsilon, \forall t \geq t_0.$$

Definition 2.2.6 *(Uniform Asymptotic Stability, [7])*

An equilibrium point x^* is said to be uniformly asymptotically stable (UAS) if:

1- Uniformly Stable (US).

2- Uniformly Attractive:

$$\exists\ \delta_a > 0 \text{ independent of } t_0, \text{ such that } \forall \varepsilon > 0, \exists T = T(\varepsilon, \delta_a) > 0,$$

$$\| x_0 - x^* \| < \delta_a \Longrightarrow \| x(t) - x^* \| < \varepsilon, \forall t \geq t_0 + T.$$

i.e

$$\exists\ \delta_a > 0 \text{ independent of } t_0, \text{ such that } \| x_0 - x^* \| < \delta_a \Longrightarrow \lim_{t \to +\infty} x(t) = x^*.$$

Definition 2.2.7 *(Exponential Stability, [7])*

An equilibrium point x^* is said to be exponentially stable (ES) if there exists a neighborhood $U(x^*)$ of x^*, and constants $\lambda_1 > 0$, $\lambda_2 > 0$ such that:

$$\| x(t) - x^* \| \leq \lambda_1 \| x_0 - x^* \| e^{-\lambda_2 (t - t_0)}, \forall x_0 \in U(x^*), \forall t \geq t_0 \geq 0.$$

Here, the constant λ_2 is called the rate of convergence (or exponential convergence rate).

Example 2.2.3 *Consider the following problem:*

$$x'(t) = -x$$
$$x(t_0) = x_0 \tag{2.2.5}$$

1- Determine the solution of problem (E).

2- Using the definition, analyze whether the equilibrium point of the problem is:

Stable, uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable.

Solution

1- The given equation is a first-order linear ODE:

$$\begin{aligned} x'(t) = -x &\Longrightarrow \frac{dx}{dt} = -x \\ &\Longrightarrow \int \frac{dx}{x} = -\int dt \\ &\Longrightarrow x(t) = Ce^{-t} \end{aligned}$$

where C is a constant determined by initial conditions.

Explicit solution:

$$x(t) = x_0 e^{-(t-t_0)}$$

2- Equilibrium Point

The system has an equilibrium point at $x = 0$, since:

$$x'(t) = 0 \Longrightarrow x = 0$$

3. Stability Analysis

a) Stable (Lyapunov Stability)

$$\forall\varepsilon > 0, \forall t_0 \geq 0, \exists\delta = \delta(t_0, \varepsilon) > 0, \textit{such that}$$
$$|x_0| < \delta \Longrightarrow |x(t)| < \varepsilon, \forall t \geq t_0.$$

we have

$$|x(t)| = \left|x_0 e^{-(t-t_0)}\right| \leq |x_0|\, e^{t_0} < \varepsilon \ \textit{Since}\ t \geq t_0 \geq 0$$

choose $\delta(t_0, \varepsilon) = \varepsilon e^{-t_0}$, *Then*

$$|x_0| < \delta \Longrightarrow |x(t)| < \varepsilon, \ \forall t \geq t_0.$$

The equilibrium is stable.

b) Uniformly Stable

$$\forall\varepsilon > 0, \exists\delta = \delta(\varepsilon) > 0, \textit{such that}\ \forall t_0 \geq 0,$$
$$\| x_0 \| < \delta \Longrightarrow \| x(t) \| < \varepsilon, \forall t \geq t_0.$$

The stability is uniform if δ *is independent of* t_0.

we have

$$|x(t)| = \left|x_0 e^{-(t-t_0)}\right| \leq |x_0| < \varepsilon \ \textit{Since}\ t \geq t_0 \geq 0$$

choose $\delta(\varepsilon) = \varepsilon$, *Then*

$$|x_0| < \delta \Longrightarrow |x(t)| < \varepsilon, \ \forall t \geq t_0.$$

The equilibrium is uniformly stable.

c) Asymptotically Stable

- Stability is already proven.

-

$$\exists\delta_a(t_0) > 0 \ \textit{such that} \ |x_0| < \delta_a \Longrightarrow \lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x_0 e^{-(t-t_0)} = 0.$$

The equilibrium is asymptotically stable.

d) Uniformly Asymptotically Stable

The convergence $x(t) \to 0$ is uniform in t_0 (i.e., the rate does not depend on t_0).

The decay rate $e^{-(t-t_0)}$ is exponential and independent of t_0.

The equilibrium is uniformly asymptotically stable.

e) Exponentially Stable

$$\exists\lambda_1, \lambda_2 > 0, \textit{such that} \ |x(t)| \leq \lambda_1 |x_0| e^{-\lambda_2(t-t_0)}, \forall t \geq t_0.$$

Here,

$$|x(t)| = \left|x_0 e^{-(t-t_0)}\right|, \ \textit{so} \ \lambda_1 = 1 \textit{and} \ \lambda_2 = 1.$$

The equilibrium is exponentially stable.

Remark 2.2.1 *(Lyapunov Stability Counterexamples, implications)*

- Lyapunov stability $\nRightarrow$ Uniform stability

  (Example: $\dot{x} = -x/(1+t)$)

- Asymptotic stability $\nRightarrow$ Uniform asymptotic stability

  (Example: $\dot{x} = -x^3/(1+t)$)

- Uniform stability + Asymptotic stability + uniform convergence $\Rightarrow$ Uniform asymptotic stability

  (Standard result - no counterexample needed)

- Exponential stability $\Rightarrow$ Asymptotic stability $\Rightarrow$ Stability

  (Example: $\dot{x} = -x$ demonstrates all three)

2.3 The case of a system of linear differential equations

Consider the differential system:

Nonhomogeneous system:

$$y'(t) = A(t)y(t) + f(t) \tag{2.3.1}$$

Homogeneous system:

$$y'(t) = A(t)y(t) \tag{2.3.2}$$

where:

- $A(t) \in M_n(\mathbb{R})$ is a continuous matrix function,
- $f(t)$ is a continuous vector function,
- defined on the interval $I = (a, +\infty)$.

Definition 2.3.1 *The system (2.3.1) is said to be* ***stable*** *(resp.* ***unstable****) if all solutions are stable (resp. unstable) in the sense of Lyapunov.*

Theorem 2.3.1 *([8]) For any arbitrary function f of class $C(I)$, the system (2.3.1) is stable if and only if the zero solution of system (2.3.2) is stable.*

Proof. ($\Rightarrow$) Let $t_0 \in I$ and $\varphi(t)$ be a stable solution of the homogeneous system (2.3.2) on the interval $[t_0, +\infty)$ with $\varphi(t_0) = y_0$. By definition of stability (in the sense of Lyapunov), for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any solution $\psi(t)$ on $[t_0, +\infty)$ with initial condition $\psi(t_0) = z_0$ satisfying $\|y_0 - z_0\| < \delta$, we have:

$$\|\varphi(t) - \psi(t)\| < \varepsilon \quad \forall t \geq t_0.$$

Now consider the function:

$$\varphi^*(t) = \varphi(t) - \psi(t) \tag{2.3.3}$$

which is a solution of the homogeneous system (2.3.2). By hypothesis, φ^* satisfies:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \|\varphi^*(t_0)\| < \delta \Rightarrow \|\varphi^*(t)\| < \varepsilon \quad \forall t \geq t_0. \tag{2.3.4}$$

Since any solution of system (2.3.2) can be written in the form (2.3.3), the condition (2.3.4) implies the stability of the zero solution of (2.3.2).

($\Leftarrow$) Conversely, assume the zero solution of (2.3.2) is stable. Then for any solution $\varphi^*(t)$ of (2.3.2) on $[t_0, +\infty)$, condition (2.3.4) holds. Let $\varphi(t)$ be a particular solution of the nonhomogeneous system (2.3.1) on $[t_0, +\infty)$ and $\psi(t)$ an arbitrary solution of the same system on $[t_0, +\infty)$. We deduce from (2.3.4) that:

$$\|\varphi(t) - \psi(t)\| < \varepsilon \quad \text{whenever} \quad \|\varphi(t_0) - \psi(t_0)\| < \delta.$$

This proves that $\varphi(t)$ is stable. ∎

Corollary 2.3.1 *([8]) The following statements hold for the differential systems (2.3.1) and (2.3.2):*

1. *The system (2.3.1) is stable (respectively asymptotically stable) if and only if the homogeneous system (2.3.2) is stable (respectively asymptotically stable).*

2. *To study the stability of solutions of system (2.3.1), it suffices to study the stability of the zero solution of system (2.3.2).*

3. *For the stability analysis of any solution of system (2.3.1), it is sufficient to study the stability of at least one of its solutions.*

4. *The system (2.3.1) is uniformly stable (respectively uniformly asymptotically stable) if and only if the zero solution of system (2.3.2) is uniformly stable (respectively uniformly asymptotically stable).*

Theorem 2.3.2 *([8]) The homogeneous system (2.3.2) is stable if and only if every solution $\varphi(t)$ of the system on the interval $[t_0, +\infty)$ is bounded.*

Proof. ($\Rightarrow$) Assume all solutions of system (2.3.2) are bounded on $[t_0, +\infty)$. This means the resolvent matrix $M(t) = \Phi(t)\Phi(t_0)^{-1}$ of system (2.3.2) is bounded, i.e., there exists a constant $K > 0$ such that:

$$\|M(t)\| \leq K \quad \forall t \geq t_0.$$

Any solution $\varphi(t)$ of system (2.3.2) can be expressed as:

$$\varphi(t) = M(t)\varphi(t_0).$$

Therefore:

$$\|\varphi(t)\| \leq \|M(t)\|\|\varphi(t_0)\| \leq K\|\varphi(t_0)\|.$$

This implies Lyapunov stability: for every $\varepsilon > 0$, choose $\delta = \varepsilon/K$ such that:

$$\|\varphi(t_0)\| < \delta \Rightarrow \|\varphi(t)\| < \varepsilon \quad \forall t \geq t_0.$$

Thus, the zero solution is stable, and consequently all solutions of system (2.3.2) are stable.

($\Leftarrow$) Conversely, suppose there exists an unbounded solution $\varphi(t)$ of system (2.3.2) on $[t_0, +\infty)$. Clearly $\varphi(t_0) \neq 0$.

For a given $\delta > 0$, define the scaled solution:

$$\psi(t) = \frac{\varphi(t)}{\|\varphi(t_0)\|} \cdot \frac{\delta}{2}.$$

Note that $\|\psi(t_0)\| = \delta/2 < \delta$.

Since $\varphi(t)$ is unbounded, there exists $t_1 > t_0$ and $\varepsilon_1 > 0$ such that:

$$\|\psi(t_1)\| = \frac{\|\varphi(t_1)\|}{\|\varphi(t_0)\|} \cdot \frac{\delta}{2} > \varepsilon_1,$$

which contradicts the stability of the zero solution of system (2.3.2). ∎

Example 2.3.1 *Consider the system:*

$$\begin{cases} x' = 1 + t - x \\ y' = 1 + t - y \end{cases} \tag{2.3.5}$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$. *Does the solution of problem (2.3.5) satisfy:*

Stable, uniformly stable, asymptotically stable, uniformly asymptotically stable?"

2.4 Well-posed and ill-posed problems

According to Jacques Hadamard [6], a mathematical problem is said to be *well-posed* (in the sense of Hadamard) if all the following conditions are satisfied:

1. A solution exists.
2. The solution is unique.
3. The solution depends continuously on the given data.

The problem is *ill-posed* if any one of these conditions is not met.

Example 2.4.1 *(Well-Posed Problems)*

The Dirichlet Problem for Laplace's Equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary.

1.
   - *Existence*: A solution exists (under smoothness conditions on f).
   - *Uniqueness*: Guaranteed by the maximum principle.

- *Stability*: Small changes in f lead to small changes in u.

Initial Value Problem for the Heat Equation

$$\begin{cases} u_t = \Delta u & \text{for } t > 0, \\ u(x,0) = f(x), \end{cases}$$

1. - *Existence & Uniqueness*: Holds under reasonable conditions on f.
   - *Stability*: The solution depends continuously on initial data f.

Example 2.4.2 *(Ill-Posed Problem) Consider the following boundary value problem:*

$$\begin{cases} -(a(x)y')' = f(x), & x \in]-1,1[\\ y(-1) = y(1) = 0 \end{cases} \tag{2.4.1}$$

Given:

- $a(x) = x^2 + 1$
- $f(x) = 3x^2 + 1$
- **Proposed solution:** $y(x) = \frac{1-x^2}{2}$

Verification:

1. **Derivative calculations:**

$$y'(x) = -x \quad \Rightarrow \quad a(x)y'^2 + 1)x$$

$$(a(x)y')' = \frac{d}{dx}\left(-x^3 - x\right) = -3x^2 - 1$$

$$-(a(x)y')'^2 + 1 = f(x) \quad \text{(verified)}$$

2. **Boundary conditions:**

$$y(-1) = \frac{1-(-1)^2}{2} = 0, \quad y(1) = \frac{1-1^2}{2} = 0 \quad \text{(verified)}$$

Ill-Posedness Observation: The differential equation admits a unique solution in this case, but the problem becomes ill-posed if:

- **Condition 1 (Existence):** For $a(x) = \frac{c}{x} + x^2 + 1$ (with $c \neq 0$), the solution doesn't exist because $a(x)$ is undefined at $x = 0$.
- **Condition 3 (Stability):** A small perturbation of $a(x)$ (e.g., $a_\epsilon(x) = x^2 + 1 + \epsilon/x$) can make the problem unsolvable.

Conclusion: Problem (2.4.1) is well-posed for $a(x) = x^2 + 1$, but slight modifications to $a(x)$ make it ill-posed (violation of existence). This illustrates sensitivity to data.

Chapter 3

Stability Analysis

Summary

This chapter studies the stability of zero solutions in different types of differential systems. First, we analyze both autonomous and non-autonomous linear systems. Then, we examine nonlinear systems using Lyapunov's two methods: the direct and indirect approaches, each applied to autonomous and non-autonomous cases.

3.1 Stability of the Zero Solution in Linear Differential Systems

3.1.1 Autonomous Linear System

Consider the system $\dot{x} = Ax$ which has the origin as an equilibrium point.

The stability properties of the origin can be characterized by the eigenvalues of the matrix A. Recall that the solution of the system is given by:

$$x(t) = e^{tA}x(0).$$

The solution of the Cauchy problem with initial condition $x(t_0) = x_0$ is given by:

$$x(t, x_0) = e^{(t-t_0)A}x_0.$$

Stability is linked to the behavior of $e^{(t-t_0)A}$ as t tends to $+\infty$, where its norm $\|e^{(t-t_0)A}\|$ must remain bounded. Let us distinguish a few cases:

First Case: Scalar System ($n = 1$)

For $A = (a) \in \mathbb{R}$, we have:

$$\left|e^{(t-t_0)a}\right| = e^{(t-t_0)\Re(a)}.$$

The solutions are stable *if and only if* this quantity remains bounded as $t \to +\infty$, which occurs when $\Re(a) \leq 0$.

Moreover, the solutions are asymptotically stable *if and only if* $\Re(a) < 0$. In this case, we have:

$$\phi(t) = e^{(t-t_0)\Re(a)} \underset{t \longrightarrow \infty}{\longrightarrow} 0.$$

Second Case: General System ($n \geq 1$)

If A is **diagonalizable**, the eigenvalues $(\lambda_i)_{1 \leq i \leq n}$ are solutions of the characteristic polynomial

$P_n(\lambda)$ of the matrix A, defined by:

$$P_n(\lambda) = \det(A - \lambda I).$$

The system reduces to independent equations:

$$x_j' = \lambda_j x_j, \quad j = 1, \ldots, n,$$

which admit the solution:

$$x_j(t; y) = y_j e^{\lambda_j (t-t_0)}, \quad j = 1, \ldots, n.$$

The solutions are:

- stable if and only if $\mathrm{Re}(\lambda_j) \leq 0$ for all j,
- asymptotically stable if and only if $\mathrm{Re}(\lambda_j) < 0$ for all j,
- unstable if there exists j such that $\mathrm{Re}(\lambda_j) > 0$.

If A is **not diagonalizable**, we analyze its Jordan canonical form. Suppose A is represented as a Jordan block:

$$A = \lambda I + N,$$

where:

- λ is an eigenvalue of A,
- N is a nilpotent matrix (strictly upper triangular) with $N^n = 0$.

The matrix exponential $e^{(t-t_0)A}$ is given by:

$$e^{(t-t_0)A} = e^{(t-t_0)\lambda I} \cdot e^{(t-t_0)N} = e^{\lambda(t-t_0)} \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} N^k.$$

- The coefficients of $e^{(t-t_0)A}$ are products of:
  - $e^{\lambda(t-t_0)}$ (exponential term),
  - Polynomials in t of degree $\leq n-1$ (due to nilpotency of N).
- Since $N \neq 0$, at least one polynomial term is non-constant (degree ≥ 1).

1. Asymptotically Stable:
   - If $Re(\lambda) < 0$, the exponential decay dominates, and all solutions tend to 0:
   $$\lim_{t\to\infty} \|e^{(t-t_0)A}\| = 0.$$
2. Unstable:
   - If $Re(\lambda) > 0$, the exponential growth dominates, and solutions diverge:
   $$\lim_{t\to\infty} \|e^{(t-t_0)A}\| = +\infty.$$
   - If $Re(\lambda) = 0$, the exponential term $|e^{\lambda(t-t_0)}| = 1$, but the polynomial terms make the solution unbounded:
   $$\|e^{(t-t_0)A}\| \text{ grows polynomially (unstable).}$$

Remark 3.1.1 *(Key Differences from the Diagonalizable Case)*

- *Non-diagonalizable case: Solutions involve polynomial terms due to nilpotency.*
- *Stability still depends on $Re(\lambda)$, but: If $Re(\lambda) = 0$, polynomial terms cause instability (unlike the diagonalizable case, where solutions remain bounded).*

Example 3.1.1 *(Stability Analysis for a Diagonalizable System)*

Consider the linear system:

$$\dot{x} = Ax, \quad \text{where} \quad A = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}.$$

The origin $x = 0$ is an equilibrium point, and A is diagonalizable.

- Eigenvalues: $\lambda_1 = -2$, $\lambda_2 = -1$.

- Eigenvectors:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

- General Solution

$$x(t) = Pe^{Dt}P^{-1}x(0), \quad e^{Dt} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

All solutions decay to 0.

- Stability

Since $Re(\lambda_i) < 0$ for all i, the origin is asymptotically stable.

Example 3.1.2 *(Non-Diagonalizable System Stability)*

Consider the system:

$$\dot{x} = Ax, \quad A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The matrix A is not diagonalizable (Jordan block).

- Jordan Decomposition

$$A = \lambda I + N, \quad \lambda = -1, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N^2 = 0.$$

- Matrix Exponential

$$e^{At} = e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- General Solution

$$x(t) = e^{-t} \begin{pmatrix} x_1(0) + t x_2(0) \\ x_2(0) \end{pmatrix}.$$

- Stability

Since $Re(\lambda) = -1 < 0$, the origin is asymptotically stable, despite the polynomial term t.

3.1.2 Non-Autonomous Linear Systems

Consider the system:

$$\dot{x} = A(t)x, \quad A(t) \text{ continuous.}$$

The solution is:

$$x(t) = R(t, t_0)x(t_0),$$

where $R(t, t_0)$ is the resolvent matrix.

Theorem 3.1.1 *(Uniform Asymptotic Stability)*

The origin is uniformly asymptotically stable iff:

$$\|R(t, t_0)\| \leq k e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0 \geq 0, \quad k, \alpha > 0.$$

Example 3.1.3 *Consider the time-varying linear system with:*

$$A(t) = \begin{pmatrix} -1 + \sin(t) & 0 \\ 0 & -1 \end{pmatrix}$$

- Resolvent Matrix Calculation

Since $A(t)$ is diagonal, the resolvent matrix $R(t, t_0)$ is:

$$R(t,t_0) = \begin{pmatrix} e^{\int_{t_0}^{t}(-1+\sin(s))ds} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix}$$

- Stability Analysis

For the first component:

$$\begin{aligned}\left|e^{\int_{t_0}^{t}(-1+\sin(s))ds}\right| &\leq e^{\int_{t_0}^{t}(-1+1)ds} \\ &= e^0 = 1 \quad (\text{since } \sin(s) \leq 1)\end{aligned}$$

The 2-norm is given by:

$$\|R(t,t_0)\| = \max\left(\left|e^{\int_{t_0}^{t}(-1+\sin(s))ds}\right|, \left|e^{-(t-t_0)}\right|\right)$$

withe

$$\begin{aligned}\left|e^{\int_{t_0}^{t}(-1+\sin(s))ds}\right| &\leq e^{\int_{t_0}^{t}(-1+1)ds} \\ &= e^0 = 1 \quad (\text{since } \sin(s) \leq 1)\end{aligned}$$

And for the second component:

$$\left|e^{-(t-t_0)}\right| = e^{-(t-t_0)} \leq 1 \quad \text{for } t \geq t_0$$

Thus:

$$\|R(t,t_0)\| \leq \max(1, e^{-(t-t_0)}) \leq 1$$

The uniform asymptotic stability requires:

$$\|R(t, t_0)\| \leq ke^{-\alpha(t-t_0)} \quad \text{with } k, \alpha > 0$$

Our computation shows:

$$\|R(t, t_0)\| \leq 1 = 1 \cdot e^{-0\cdot(t-t_0)}$$

This corresponds to $\alpha = 0$, which fails the stability requirement ($\alpha > 0$ needed).

- Modified System for Stability

Consider the modified system:

$$A(t) = \begin{pmatrix} -2 + \sin(t) & 0 \\ 0 & -1 \end{pmatrix}$$

Now the integral becomes:

$$\int_{t_0}^{t} (-2 + \sin(s))ds \leq -1.5(t - t_0) \quad (\text{since } \sin(s) \geq -1)$$

Thus:

$$\|R(t, t_0)\| \leq e^{-1.5(t-t_0)}$$

The modified system satisfies:

$$\|R(t, t_0)\| \leq ke^{-\alpha(t-t_0)}$$

with $k = 1$ and $\alpha = 1.5 > 0$, proving uniform asymptotic stability.

3.2 Stability of the Zero Solution in Nonlinear Differential System

3.2.1 Autonomous nonLinear System (Lyapunov's direct method)

Consider a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ and its associated autonomous system:

$$\dot{x} = f(x) \tag{3.2.1}$$

Definition 3.2.1 (Total Derivative) *For the system (3.2.1) and a function $V : \mathbb{R}^n \to \mathbb{R}$ with partial derivatives on $\mathbb{R}^n$, we define the total derivative $\dot{V}$ of the system (3.2.1) as:*

$$\dot{V}(y) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(y) f_i(y) \tag{3.2.2}$$

where $f_i(y)$ are the components of $f(y) = (f_1(y), \ldots, f_n(y))^\top$.

Definition 3.2.2 *A function $v : \mathbb{R}^n \to \mathbb{R}$ is said to be:*

- *Positive semi-definite (resp. negative semi-definite) if there exists a neighborhood $\mathcal{V}$ of 0 such that:*

  1. $v(0) = 0$
  2. $\forall y \in \mathcal{V},\ v(y) \geq 0$ *(resp. $v(y) \leq 0$)*

- *Positive definite (resp. negative definite) if there exists a neighborhood $\mathcal{V}$ of 0 such that:*

  1. $v(0) = 0$
  2. $\forall y \in \mathcal{V} \setminus \{0\},\ v(y) > 0$ *(resp. $v(y) < 0$)*

Theorem 3.2.1 *([7]) Consider the autonomous system (3.2.1) with the origin as an equilibrium point. If there exists a neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of 0 and a continuous function $V : \mathcal{V} \to \mathbb{R}^+$ with continuous partial derivatives such that:*

1. *V is positive definite*

2. *The total derivative $\dot{V}$ of (3.2.1) is negative semi-definite*

then the origin is stable.

Moreover, if $\dot{V}$ is negative definite, then:

- *The origin is asymptotically stable.*

V is called a Lyapunov function.

Example 3.2.1 *Consider the dynamical system:*

$$\begin{cases} x_1' = -x_1^3 - x_2^2 \\ x_2' = x_1 x_2 - x_2^3 \end{cases}$$

To analyze the stability of the equilibrium point at the origin $\mathbf{0}$, we propose the Lyapunov function candidate:

$$V(y_1, y_2) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2$$

- Lyapunov Function Properties

- $V(\mathbf{0}) = 0$

- *V is positive definite since it is quadratic and strictly positive for all $(y_1, y_2) \neq (0, 0)$*

- Time Derivative Analysis The total derivative of V along system trajectories is:

$$\begin{aligned} \dot{V}(y_1, y_2) &= \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 \\ &= y_1(-y_1^3 - y_2^2) + y_2(y_1 y_2 - y_2^3) \\ &= -y_1^4 - y_1 y_2^2 + y_1 y_2^2 - y_2^4 \\ &= -(y_1^4 + y_2^4) \end{aligned}$$

- Stability Conclusion

- *$\dot{V}$ is negative definite since $y_1^4 + y_2^4 > 0$ for all $(y_1, y_2) \neq (0,0)$*
- *According to Theorem 3.2.1 (Lyapunov's Direct Method), the origin is asymptotically stable*

3.2.2 Non-Autonomous nonLinear Systems (Lyapunov's direct method)

Consider a continuous function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ (where $I \subseteq \mathbb{R}$) and its associated non-autonomous system:

$$\dot{x} = f(t, x) \tag{3.2.3}$$

Definition 3.2.3 (Total Derivative) *For the system (3.2.3) and a function $V : I \times \mathbb{R}^n \to \mathbb{R}$ with partial derivatives on $I \times \mathbb{R}^n$, we define the total derivative $\dot{V}$ as:*

$$\dot{V}(t, y) = \frac{\partial V}{\partial t}(t, y) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, y) f_i(t, y) \tag{3.2.4}$$

where $f(t, y) = (f_1(t, y), \ldots, f_n(t, y))^\top$ represents the component functions.

Definition 3.2.4 *A function $v : I \times \mathbb{R}^n \to \mathbb{R}$ is said to be*

- *positive semi-definite (resp. negative semi-definite) if there exists a neighborhood $\mathcal{V}$ of 0 such that:*

1. $\forall t \in I,\ v(t, 0) = 0$
2. $\forall t \in I, \forall y \in \mathcal{V},\ v(t, y) \geq 0$ *(resp. $v(t, y) \leq 0$)*

- *positive definite (resp. negative definite) if there exists a neighborhood $\mathcal{V}$ of 0 such that:*

1. $\forall t \in I,\ v(t, 0) = 0$
2. *There exists a positive definite function $v_0 : \mathcal{V} \to \mathbb{R}$ such that:*

$$\forall t \in I, \forall y \in \mathcal{V},\ v(t, y) \geq v_0(y)$$

(resp. $v(t, y) \leq v_0(y)$)

Definition 3.2.5 *A function $v : I \times \mathbb{R}^n \to \mathbb{R}$ is called decrescent if:*

1. $\lim_{\|y\|\to 0} v(t, y) = 0$

   uniformly in t, which means:

2. $\forall \epsilon > 0, \exists \delta > 0$ *such that* $\forall y \in \mathbb{R}^n$*:*

$$\|y\| < \delta \Rightarrow \forall t \in I, |v(t, y)| < \epsilon$$

Theorem 3.2.2 *([7]) Let the origin be an equilibrium point of the non-autonomous system (3.2.3). If there exists a neighborhood $\mathcal{V}_{t_0} \subset \mathbb{R}^n$ of 0 and a continuous function $V : \mathcal{V}_{t_0} \to \mathbb{R}^+$ with continuous partial derivatives such that:*

1. *V is positive definite*

2. *The total derivative $\dot{V}$ for (3.2.3) is negative semi-definite (resp. negative definite)*

then the origin is uniformly stable.

Furthermore, if:

1. *V is decrescent, then the origin is uniformly stable (resp. uniformly asymptotically stable).*

Consider the non-autonomous dynamical system:

$$\begin{cases} x_1' = -x_1 - e^{-2t}x_2 \\ x_2' = x_1 - x_2 \end{cases}$$

We propose the following time-dependent Lyapunov function candidate:

$$V(t, y) = y_1^2 + (1 + e^{-2t})y_2^2$$

- V is positive definite because it dominates the time-independent positive definite function:

$$V_0(y) = y_1^2 + y_2^2$$

since $V(t, y) \geq V_0(y)$ for all $t \geq 0$ and all $y \in \mathbb{R}^2$

- Specifically:

$$\begin{aligned} V(t, y) &= y_1^2 + (1 + e^{-2t})y_2^2 \\ &\geq y_1^2 + y_2^2 \quad (\text{since } 1 + e^{-2t} \geq 1 \text{ for } t \geq 0) \end{aligned}$$

- V is decrescent because it is dominated by the time-independent positive definite function:

$$V_1(y) = y_1^2 + 2y_2^2$$

since $V(t, y) \leq V_1(y)$ for all $t \geq 0$ and all $y \in \mathbb{R}^2$

- Specifically:

$$\begin{aligned} V(t, y) &= y_1^2 + (1 + e^{-2t})y_2^2 \\ &\leq y_1^2 + 2y_2^2 \quad (\text{since } 1 + e^{-2t} \leq 2 \text{ for } t \geq 0) \end{aligned}$$

The total derivative of V along system trajectories is:

$$\begin{aligned} \dot{V}(t, y) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 \\ &= -2e^{-2t}y_2^2 + 2y_1(-y_1 - e^{-2t}y_2) + 2(1 + e^{-2t})y_2(x_1 - x_2) \\ &= -2y_1^2 - 2e^{-2t}y_1y_2 - 2e^{-2t}y_2^2 + 2(1 + e^{-2t})y_1y_2 - 2(1 + e^{-2t})y_2^2 \\ &= -2y_1^2 + 2y_1y_2 - 2y_2^2 - 2e^{-2t}y_2^2 \\ &= -2(y_1^2 - y_1y_2 + y_2^2(1 + e^{-2t})) \end{aligned}$$

We can establish the following bounds:

$$\begin{aligned}\dot{V}(t,y) &\leq -2(y_1^2 - y_1y_2 + y_2^2)\\ &= -2\left[\left(y_1 - \frac{y_2}{2}\right)^2 + \frac{3}{4}y_2^2\right]\\ &\leq -(y_1 - y_2)^2 - y_1^2 - y_2^2\end{aligned}$$

Stability Conclusion

- $\dot{V}$ is negative definite (all terms are strictly negative for all $y \neq 0$)
- V is positive definite and decrescent
- $\dot{V}$ is negative definite
- Therefore, by Theorem 3.2.2, the origin $\mathbf{0}$ is uniformly asymptotically stable

3.2.3 Autonomous nonLinear System (Lyapunov's indirect method)

The study of stability in nonlinear systems is challenging due to their complex behavior, as linear methods are no longer applicable. However, Lyapunov and others observed that, in most cases, the equilibrium points of nonlinear systems can be classified similarly to those of linear systems by analyzing integral curve trajectories near equilibrium.

Lyapunov's indirect method leverages this idea by examining stability through linearization. The nonlinear system is approximated as:

$$\frac{dx}{dt} = Df(x_e)x + O(\|x\|^2),$$

where $Df(x_e)$ is the Jacobian matrix evaluated at the equilibrium point. This linearization converts the global stability problem into a local one near the equilibrium point.

Stability Analysis via Linear Approximation To address stability questions, we consider the linear system associated with (3.2.1) :

$$\frac{dx}{dt} = Df(x_e)x$$

where $Df(x_e)$ is the Jacobian matrix of f evaluated at the equilibrium point x_e:

$$A = Df(x_e) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{x=x_e}$$

The determination of the stability of the equilibrium point is therefore carried out in two steps:

Step 1: Linear Stability Analysis

1. Linearized System:

   Transform the nonlinear system (3.2.1) into its linear approximation about equilibrium x_e:

   $$\frac{dx}{dt} = Ax \quad \text{where} \quad A = Df(x_e)$$

   Here $Df(x_e)$ is the Jacobian matrix evaluated at x_e.

2. Eigenvalue Criterion:

   Determine stability of $x = 0$ in the linear system by examining eigenvalues λ_i of A:

   - $\Re(\lambda_i) < 0$ for all $i \Rightarrow$ asymptotically stable
   - $\exists \lambda_i$ with $\Re(\lambda_i) > 0 \Rightarrow$ unstable
   - $\Re(\lambda_i) = 0$ for some $i \Rightarrow$ critical case (indeterminate)

Step 2: Nonlinear Stability Equivalence

1. Stability Transfer:

   Establish conditions under which linear stability implies nonlinear stability:

   - If linear system is stable $\Rightarrow x_e$ is *locally* stable for (3.2.1)
   - If linear system is unstable $\Rightarrow x_e$ is unstable for (3.2.1)

2. Critical Case Handling:

   When $\Re(\lambda_i) = 0$, linearization is insufficient. Requires:

   - Higher-order expansion, or
   - Direct Lyapunov methods

Theorem 3.2.3 (Hartman-Grobman) *[1] Consider the dynamical system (3.2.1) with flow ϕ_t. If x_e is a hyperbolic equilibrium point, then there exists a neighborhood $\mathcal{V}$ of x_e where the flow ϕ_t is topologically conjugate to the flow of the linearized system at x_e. Consequently, we have the following result:*

Immediate Corollary. The topological equivalence implies that:

- The stability properties of the nonlinear system match those of its linearization
- The phase portrait near x_e is homeomorphic to the linearized system's portrait
- All hyperbolic equilibria are structurally stable

■

Theorem 3.2.4 (Lyapunov's Linearization Method) *[7] For the nonlinear system $\dot{x} = f(x)$ with equilibrium x_e (i.e., $f(x_e) = 0$), and its linearization $\dot{x} = Df(x_e)x$, the following equivalence holds:*

1. *If the linearized system is asymptotically stable (all $\Re(\lambda_i) < 0$), then x_e is locally asymptotically stable for the nonlinear system.*

2. *If the linearized system is unstable ($\exists\lambda_i$ with $\Re(\lambda_i) > 0$), then x_e is unstable for the nonlinear system.*

3. *If the linearized system is stable ($\Re(\lambda_i) \leq 0$), then x_e is stable for the nonlinear system.*

Example 3.2.2 *Consider the system:*

$$\frac{dx}{dt} = \sin x - x$$

- *Equilibrium:* $x_e = 0$
- *Jacobian:* $Df(0) = \cos(0) - 1 = 0$
- *Eigenvalue:* $\lambda = 0$ *(critical case)*

Conclusion: Linearization fails - requires center manifold or direct Lyapunov analysis.

Example 3.2.3 *Consider the nonlinear system:*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a \sin x_1 - b x_2 \end{cases}$$

The system has two equilibrium points:

Origin: $(x_1, x_2) = (0, 0)$

Second point: $(x_1, x_2) = (\pi, 0)$

The Jacobian matrix of the system is:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a \cos x_1 & -b \end{bmatrix}$$

Stability at the Origin (0,0)

Evaluating the Jacobian at the origin:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The characteristic equation is:

$$\det(\lambda I - A) = \lambda^2 + b\lambda + a = 0$$

Eigenvalues are:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a}}{2}$$

Stability Conditions

For $a, b > 0$:

When $b^2 > 4a$ (real distinct roots):

$$\lambda_{1,2} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4a}}{2} < 0$$

When $b^2 < 4a$ (complex conjugate roots):

$$\Re(\lambda_{1,2}) = -\frac{b}{2} < 0$$

Critical damping ($b^2 = 4a$):

$$\lambda = -\frac{b}{2} < 0$$

Stability at (π,0)

Evaluating the Jacobian at $(\pi, 0)$:

$$\tilde{A} = \frac{\partial f}{\partial x}\bigg|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

The characteristic equation is:

$$\det(\lambda I - \tilde{A}) = \lambda^2 + b\lambda - a = 0$$

Eigenvalues are:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 + 4a}}{2}$$

For $a > 0$ and $b \geq 0$:

$\lambda_1 = -\frac{b}{2} + \frac{\sqrt{b^2+4a}}{2} > 0$ *(since $\sqrt{b^2 + 4a} > b$)*

$\lambda_2 = -\frac{b}{2} - \frac{\sqrt{b^2+4a}}{2} < 0$

Phase Portrait Analysis

- Near (0,0): All trajectories converge to the origin (stable node/focus)

- Near (π,0):

1- One unstable direction (along eigenvector of $\lambda_1 > 0$)2- One stable direction (along eigenvector of $\lambda_2 < 0$)

3.2.4 Non-Autonomous nonLinear Systems (Lyapunov's indirect method)

For this section (see [3], (Chapter 4 on non-autonomous systems, [7])).

We consider the system:

$$y' = f(t, y) = A(t)y + F(t, y)$$

where:

- $A(t)$ is a non-constant time-dependent matrix
- $F(t, y)$ represents nonlinear terms with $\lim_{\|y\| \to 0} \frac{\|F(t,y)\|}{\|y\|} = 0$

When $A(t)$ is not constant, we cannot directly use eigenvalues to study stability. More advanced tools are required.

Linearized System Analysis

$$y' = A(t)y$$

Example 3.2.4 (Counterintuitive Case) *Consider:*

$$A(t) = \begin{pmatrix} -1 + \alpha\cos^2 t & 1 - \alpha\sin t\cos t \\ -1 - \alpha\sin t\cos t & -1 + \alpha\sin^2 t \end{pmatrix}$$

- *Eigenvalues:* $\lambda = -1$ *(constant and stable)*
- *However, for* $\alpha > 1$*, the system is unstable*

$\Rightarrow$ *Eigenvalues alone are insufficient.*

Theoretical Tools

Theorem 3.2.5 (Floquet's Theorem (Periodic Case)) *If* $A(t+T) = A(t)$*, then:*

- *Stability is determined by Floquet multipliers*
- *Stability condition:* $|\mu_i| < 1 \Rightarrow$ *exponential stability*

Theorem 3.2.6 (Lyapunov's Method) *Construction of a function* $V(t,y)$ *such that:*

1. $V(t,y)$ *is positive definite*
2. $\dot{V}(t,y) = \frac{\partial V}{\partial t} + \nabla V \cdot f(t,y)$ *is negative definite*

$\Rightarrow y = 0$ *is uniformly asymptotically stable.*

General Stability Analysis Strategy

1. Analyze the linearized system $y' = A(t)y$:
   - If periodic $\rightarrow$ Floquet theory
   - Otherwise $\rightarrow$ Time-varying Lyapunov function
2. Control nonlinear terms: $\|F(t,y)\| \leq L\|y\|^{1+\alpha}$

3. Draw stability conclusions

Example 3.2.5

$$y' = \begin{pmatrix} -2+\sin t & 1 \\ 0 & -3+\cos t \end{pmatrix} y + \begin{pmatrix} y_1^2 \\ y_1 y_2 \end{pmatrix}$$

- *Instantaneous eigenvalues:* $\lambda_1(t) = -2+\sin t$, $\lambda_2(t) = -3+\cos t$
- *Solution approach: Use* $V(t,y) = y_1^2 + (1+e^{-t})y_2^2$

Conclusion

- Instantaneous eigenvalues are insufficient
- Required tools:
  - Time-varying Lyapunov functions
  - Floquet theory
  - Numerical methods (e.g., Lyapunov exponents)
- Nonlinear terms must be dominated by the linear part

Chapter 4

Tutorials

This chapter provides review exercises covering previous chapters along with selected problems from past examinations.

4.1 Tuts (Reminders and Fundamental Concepts)

Exercise 4.1.1 *A) Find solutions to the following homogeneous problems:*

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}, \begin{cases} y' = \frac{1}{t}y \\ y(1) = \pi \end{cases}, \begin{cases} y' = e^t y \\ y(0) = e \end{cases}, \begin{cases} y' = \frac{t}{\sqrt{4-t^2}}y \\ y(2) = 0 \end{cases},$$
$$\begin{cases} y' = \ln(t)\, y \\ y(1) = 1 \end{cases}, \begin{cases} y' = \sin(t)\cos(t)\, y \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$$

B) Solve the following Cauchy problems:

$$1) \begin{cases} y' = -5y + 3 \\ y(0) = 0 \end{cases}, \quad 2) \begin{cases} y' = -3y + 4e^t \\ y(0) = -2 \end{cases},$$
$$3) \begin{cases} y' = 3y + \sin(3t) + \sin(2t) \\ y(0) = 0 \end{cases}, 4) \begin{cases} y' \sin(t) - y\cos(t) = 3t^2 \sin^2 t \\ y'(0) = 2 \end{cases}$$

C) Solve the following inhomogeneous problems:

$$\left(1 + t^2\right) y' = 2ty + \left(1 + t^2\right)^2, \; y' + 2ty = 2te^{-t^2}, \; y' = t^2(1 - y).$$

D) Solve the following Cauchy problems:

$$\begin{cases} y' = y + y^2 \\ y(0) = 1 \end{cases}, \begin{cases} y' = \frac{t}{1+t^2}y + ty^2 \\ y(0) = -3 \end{cases}, \begin{cases} y' = y - 2ty^3 \\ 2y\left(\frac{1}{2}\right) = \sqrt{e} \end{cases}.$$

Exercise 4.1.2 *Consider the following Cauchy problem:*

$$(E) \begin{cases} y'(t) = \sqrt{|y(t)|} \\ y(0) = 0 \end{cases}, \quad t \in \mathbb{R}. \tag{4.1.1}$$

1. *Verify that* $y(t) = 0$ *is a solution of (4.1.1).*

2. *Construct a non-zero solution of class C^1 for this problem.*

3. *Does the Cauchy-Lipschitz theorem apply here? Why?*

Exercise 4.1.3 *(Gronwall's Lemma - Integral Form) Let $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $d \geq 0$, and let $\Psi \in C^0([a, b])$. Suppose that:*

$$\Psi(t) \leq c + d \int_a^t \Psi(u)du \quad \text{for all } t \in [a, b]$$

Consider the function h defined on $[a, b]$ by:

$$h(t) = c + d \int_a^t \Psi(u)du$$

Note that h is a function defined via an integral.

1. *Show that $h \in C^1([a, b])$ and compute its derivative.*

2. *Show that for all $t \in [a, b]$, we have $h'(t) \leq dh(t)$. Then deduce that:*

$$h(t) \leq ce^{d(t-a)} \quad \text{for all } t \in [a, b]$$

3. *Deduce that:*

$$\Psi(t) \leq ce^{d(t-a)} \quad \text{for all } t \in [a, b]$$

Exercise 4.1.4 *(Gronwall's Lemma - Differential Form) Let $\eta : [a, b] \to \mathbb{R}_+$ be a continuous function satisfying the differential inequality:*

$$\eta'(t) \leq \Phi(t)\eta(t) + \psi(t) \quad \text{for almost every } t \in [a, b],$$

where $\Phi, \psi : [a, b] \to \mathbb{R}_+$ are non-negative functions.

Show that for all $t \in [a, b]$, the following inequality holds:

$$\eta(t) \leq \exp\left(\int_a^t \Phi(s)ds\right) \left[\eta(a) + \int_a^t \psi(s)ds\right] \quad \forall t \in [a, b].$$

In particular, if:

$$\begin{cases} \eta'(t) \leq \Phi(t)\eta(t) \quad \text{on } [a,b], \\ \eta(a) = 0, \end{cases}$$

prove that $\eta \equiv 0$ *on* $[a,b]$.

Exercise 4.1.5 *Let* $y : J \to \mathbb{R}$ *be a function. Show that* y *is a solution to the Cauchy problem (P) if and only if the following two conditions are satisfied:*

(i) y *is continuous on* J *and for all* $t \in J$, $(t, y(t)) \in J \times \Omega \subseteq I \times \Omega$,

(ii) *For all* $t \in J$, $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$.

Exercise 4.1.6 *Let* $t, y \in \mathbb{R}$ *with* $ty \neq 0$, *consider the differential equation:*

$$y' = t\sqrt{t^2 + y^2} = f(t, y), \tag{4.1.2}$$

1. *Show that:*

$$|f(t, y)| \leq |t|^2 + |t||y|$$

2. *Are all maximal solutions of equation (4.1.2) global?*

Exercise 4.1.7 *Show that the Cauchy problem (4.1.3):*

$$\begin{cases} y' = t + y^3 \\ y(0) = 0 \end{cases} \tag{4.1.3}$$

admits a unique local solution defined on the interval $[-T, T]$, *where* T *is given by the Cauchy-Lipschitz theorem.*

Exercise 4.1.8 ***(Assignment)*** *Consider the following differential equation:*

$$\begin{cases} y' + \left(\frac{2t}{1+t^2}\right) y + t^2 y^2 = 0 \\ y(0) = 1 \end{cases} \tag{4.1.4}$$

1. *Show that equation (4.1.4) admits a unique maximal solution* $\phi : I = (-T, +T) \to \mathbb{R}$ *of class* C^1 *with* $-T < 0 < +T$.

2. *Verify that* ϕ *is of class* C^2 *on* I. *Is it of class* C^1*?*

3. *Show that* $\phi(t) > 0$ *for all* $t \in I$.

4. *Deduce that* ϕ *is strictly decreasing on* $[0, T^+)$ *and that:*

$$0 < \phi(t) \leq 1, \quad \forall t \in [0, T^+)$$

Exercise 4.1.9 *(**Assignment**) Let* $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ *be the function defined by:*

$$\forall t \in \mathbb{R}, \forall y > 0, \quad f(t, y) = -\frac{1}{y}$$

1. *Show that* f *is continuous on* $\mathbb{R} \times (0, +\infty)$ *and locally Lipschitz with respect to its second variable.*

2. *For any* $y_0 > 0$, *consider the Cauchy problem:*

$$\begin{cases} y'(t) = -\dfrac{1}{y(t)} \\ y(0) = y_0 \end{cases}, \quad t \in \mathbb{R}. \tag{4.1.5}$$

3. *Explicitly compute the maximal solution to problem (4.1.5) on its interval of definition* (t_*, T^*), *where* t_* *and* T^* *may be finite or infinite.*

4. *Study the behavior of* $y(t)$ *as* $t \to T^*$. *Relate this to the theorem of the endpoints.*

Exercise 4.1.10 *(**Assignment**) Consider the following differential equation:*

$$(E) \begin{cases} y'(t) = \sin(y(t)) \\ y(0) = y_0 \in \mathbb{R} \end{cases} \tag{4.1.6}$$

1. *Show that equation (4.1.6) admits a unique maximal solution* y.

2. *Show that this solution is global.*

3. *Furthermore, show that this solution is of class* C^∞.

4. *What are the stationary solutions of (4.1.6)?*

5. *Assume that* $0 < y_0 < \pi$. *Show that:* $\forall t \in \mathbb{R}, \quad 0 < y(t) < \pi$

4.2 Tuts (The Concept of Stability)

Exercise 4.2.1 *Consider the following problem:*

$$\begin{cases} x'(t) = \dfrac{x(t)}{t^2} \\ x(t_0) = x_0, \quad t_0 > 0 \end{cases} \tag{4.2.1}$$

1. *Determine the solution of problem (4.2.1).*

2. *Using the definition, analyze whether the equilibrium point of the problem is:*

   - *Stable*
   - *Uniformly stable*
   - *Asymptotically stable*
   - *Uniformly asymptotically stable*

Exercise 4.2.2 *Consider the following problem:*

$$\begin{cases} x'(t) = -\dfrac{x(t)}{1+t} \\ x(t_0) = x_0, \quad t \in]-1, +\infty[\end{cases} \tag{4.2.2}$$

1. *Determine the solution of problem (4.2.2).*

2. *Using the definition, analyze whether the equilibrium point of the problem is:*

   - *Stable*
   - *Uniformly stable*
   - *Asymptotically stable*
   - *Uniformly asymptotically stable*

Exercise 4.2.3 *Consider the following initial value problem:*

$$\begin{cases} x'(t) = x(t) \\ x(t_0) = x_0 \end{cases} \tag{4.2.3}$$

Show that a zero solution of the problem is unstable.

Exercise 4.2.4 *Consider the following initial value problem:*

$$\begin{cases} x'(t) = (6t\sin(t) - 2t)\,x \\ x(t_0) = x_0 \end{cases} \tag{4.2.4}$$

1. *Determine the solution of problem (4.2.4).*

2. *Using the definition, analyze whether the equilibrium point of the problem is:*

   - *Stable*
   - *Uniformly stable*
   - *Asymptotically stable*
   - *Uniformly asymptotically stable*

Exercise 4.2.5 *Consider the following problem:*

$$\begin{cases} x'(t) = 2y \\ y'(t) = -x - 3y \\ x(0) = x_0,\ y(0) = y_0 \end{cases} \tag{4.2.5}$$

1. *Determine the solution of problem (4.2.5).*
2. *Using the definition, analyze whether the equilibrium point of the problem is:*
   - *Stable*
   - *Uniformly stable*
   - *Asymptotically stable*
   - *Uniformly asymptotically stable*

4.3 Tuts (Stability Analysis)

Exercise 4.3.1 *Determine the nature of the rest points for the following systems:*

1. $$\begin{cases} x_1' = 3x_1 \\ x_2' = 2x_2 \end{cases} \quad \textit{where} \quad A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$
2. $$\begin{cases} x_1' = -2x_1 + x_2 \\ x_2' = x_1 - 2x_2 \end{cases} \quad \textit{where} \quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$
3. $$\begin{cases} x_1' = x_1 \\ x_2' = -2x_2 \end{cases} \quad \textit{where} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

4.

$$\begin{cases} x_1' = x_1 \\ x_2' = -2x_2 \end{cases} \qquad where \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

5.

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_2 \end{cases} \qquad where \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Exercise 4.3.2 *Determine the nature of the rest points for the following systems:*

1.

$$\begin{cases} x_1' = x_1(2x_2 - 1) \\ x_2' = x_2(2x_1 - 1) \end{cases}$$

2.

$$\begin{cases} x' = x_2(x_1 + 1) \\ y' = x_1(x_2^3 + 1) \end{cases}$$

Exercise 4.3.3 *Use the Lyapunov function method to study the stability of the system:*

$$\begin{cases} x_1' = x_2 + x_1(x_1^2 + x_2^2) \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$$

(Hint: Consider the function $V(x_1, x_2) = x_1^2 + x_2^2$.)

Exercise 4.3.4 *Use the Lyapunov function method to study the stability of the system:*

$$\begin{cases} x_1' = -x_2 - x_1^3 \\ x_2' = x_1 - x_2^3 \end{cases}$$

(Hint: Consider the function $V(x, y) = x^2 + y^2$.)

Exercise 4.3.5 *Consider the following system:*

$$\begin{cases} x_1' = -x_2^3, \\ \\ x_2' = x_1. \end{cases} \tag{4.3.1}$$

1. *Can we study the stability of system (4.3.1) using the linearization method?*

2. *Consider the function* $V(x, y) = 2x^2 + y^4$. *Study the stability of system (4.3.1).*

4.4 The controls

(FINAL EXAM 2022)

Exercise 4.4.1 *Find the set of maximal solutions in* $\mathbb{R}$ *to the differential equation:*

$$|t|\, x' + x = t^2$$

Exercise 4.4.2 *Consider the following system:*

$$(E) \quad \begin{cases} x_1'(t) = 2x_1x_2 + \varepsilon x_1^3 - 2\omega x_2, \\ x_2'(t) = -x_1^2 + \varepsilon x_2^5 + \omega x_1, \end{cases} \qquad \text{where } \omega \in \mathbb{R}, \varepsilon < 0.$$

Given the Lyapunov function:

$$V(y_1, y_2) = y_1^2 + 2y_2^2,$$

answer the following:

1. *Is the origin an equilibrium point of system* (E)*?*
2. *Is the origin stable?*
3. *Is the origin asymptotically stable?*

Exercise 4.4.3 *Consider the following problem:*

$$(P) \quad \begin{cases} X'(t) = -\left(\frac{1}{1+t}\right) X, \\ X(t_0) = X_0, \end{cases} \qquad t \in]0, +\infty[.$$

1. *Determine the solution of* (P).
2. *Using the definition, determine whether the origin is:*
   - *Stable,*

- *Asymptotically stable,*
- *Uniformly stable,*
- *Uniformly asymptotically stable.*

(FINAL EXAM 2023)

Exercise 4.4.4 *Consider the following system:*

$$\begin{cases} x_1'(t) = -x_1 + x_2 \\ x_2'(t) = -x_1 - x_2 \end{cases} \qquad (E)$$

Consider the following Lyapunov function:

$$V(y_1, y_2) = y_1^2 + y_2^2$$

1. *Write (E) in matrix form* $X' = AX$.
2. *Is the origin an equilibrium point of system (E)?*
3. *Use two different methods to study the stability of system (E).*
4. *If* λ_1 *is a complex eigenvalue of* A *such that* $Im(\lambda_1) < 0$ *and* V_1 *is its associated eigenvector, determine the two functions:*

$$X_1(t) = Re(V_1 e^{\lambda_1 t}), \quad X_2(t) = Im(V_1 e^{\lambda_1 t})$$

5. *Show that* $\{X_1, X_2\}$ *forms a fundamental system of solutions for (E).*
6. *Find the fundamental matrix of (E).*
7. *Find the general solution of system (E).*
8. *Deduce the solution of (E) with initial conditions* $x_1(0) = x_0$, $x_2(0) = y_0$.

9. *Using the definition, determine whether the origin is stable, asymptotically stable.*

Exercise 4.4.5 *Show that for all $(x_0, y_0) \in \mathbb{R}^2$, the Cauchy problem*

$$\begin{cases} y_1'(t) = y_2 \\ y_2'(t) = -(1 - y_1^2)y_2 - y_1 - y_1^2 \\ y_1(0) = x_0, \quad y_2(0) = y_0 \end{cases}$$

admits a unique maximal solution, defined on an open interval I.

Consider the following system:

$$\begin{cases} y_1'(t) = y_2 \\ y_2'(t) = -(1 - y_1^2)y_2 - y_1 - y_1^2 \end{cases} \qquad (E)$$

1. *Show that system (E) has exactly two equilibrium points, and determine them.*

2. *Study the stability of these two equilibrium points.*

(FINAL EXAM 2024)

Exercise 4.4.6 *For each of the following statements, answer True or False with justification:*

1. *The differential equation $\frac{\partial^2 y}{\partial t^2} + \sin(t)\frac{\partial y}{\partial t} + 1 = 0$ is partial, nonlinear, with constant coefficients.*

2. *If z is a solution of $y' = f(t, y)$ that is not maximal, then it is not global.*

3. *The problem (E)*

$$\begin{cases} y' = y + t^2 y^3 \\ y(0) = 0 \end{cases}$$

admits a unique maximal solution ϕ. This solution is of class C^3.

4. *Every maximal solution of the equation* $y' = \sqrt{t}e^{-t} + \frac{y}{1+y^2}\cos(\sqrt{y})$ *is global.*

5. *The solution of the differential equation* $ty\prime - 2y - t^3e^t = 0$ *is given by* $y(t) = t^2e^t + ct^2$.

6. *For*

$$Y_1(t) = \begin{pmatrix} e^{\frac{t^2}{2}+t} \\ -e^{\frac{t^2}{2}+t} \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} e^{\frac{t^2}{2}-t} \\ -2e^{\frac{t^2}{2}-t} \end{pmatrix}$$

forms a fundamental system of solutions for (H)

$$Y'(t) = A(t)Y(t), \quad \textit{where} \quad A(t) = \begin{pmatrix} t+3 & 2 \\ -4 & t-3 \end{pmatrix} \quad \textit{(H)}$$

The resolvent matrix of (H) is given by

$$R(t,0) = \begin{pmatrix} 2e^{\frac{t^2}{2}+t} - e^{\frac{t^2}{2}-t} & e^{\frac{t^2}{2}+t} - e^{\frac{t^2}{2}-t} \\ -2e^{\frac{t^2}{2}+t} + 2e^{\frac{t^2}{2}-t} & -e^{\frac{t^2}{2}+t} + 2e^{\frac{t^2}{2}-t} \end{pmatrix}$$

7. *Consider the system* $X' = f(X)$, *where* $f(0) = 0$ *and there exists a* C^1 *function* V *such that:*

$$V : U \subset \mathbb{R}^n \to \mathbb{R}, \quad \textit{with } U \textit{ a neighborhood of } 0$$

   (a) *If* $V(0) = 0$, $V(X) > 0$ *for all* $X \neq 0$, *and* $\dot{V}(X) = \frac{d}{dt}V(X) < 0$, *then the origin is stable.*

   (b) *If* $\dot{V}(X) = \frac{d}{dt}V(X) > 0$, *then the origin is unstable.*

   (c) *If the origin is stable, then it is asymptotically stable.*

Exercise 4.4.7 *Consider the following system:*

$$\begin{cases} x_1'(t) = x_2 + x_1(x_1^2 + x_2^2) \\ x_2'(t) = -x_1 + x_1(x_1^2 + x_2^2) \end{cases} \quad (E)$$

Consider the following function:

$$V(y_1, y_2) = y_1^2 + y_2^2$$

1. *Show that the origin is an equilibrium point of system (E).*
2. *Study the stability of the equilibrium point of system (E) using the Lyapunov method.*
3. *Study the stability of the equilibrium point of system (E) using the linearization method.*

(FINAL EXAM 2025)

Exercise 4.4.8 *The three questions are independent.*

I) *List 3 methods for studying stability.*

II) *Let $a \in \mathbb{R}^*$. Study the stability of the system:*

$$Y' = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} Y$$

III)

1. *Show that for all $(x_0, y_0) \in \mathbb{R}^2$, the Cauchy problem*

$$\begin{cases} x' = y - x\,(x^2 + y^2) \\ y' = -x - y\,(x^2 + y^2) \\ x(0) = x_0, \quad y(0) = y_0 \end{cases}$$

   admits a unique maximal solution, defined on an open interval I.

2. *Consider the Lyapunov function defined on $\mathbb{R}^2$ by $V(x, y) = x^2 + y^2$ and the system:*

$$\begin{cases} x' = y - x\,(x^2 + y^2) \\ y' = -x - y\,(x^2 + y^2) \end{cases}$$

(a) Is the origin an equilibrium point of the system?

(b) Use two methods to study the stability of the origin for this system.

Exercise 4.4.9 *Consider the following system:*

$$\begin{cases} x_1'(t) = 2x_2 \\ x_2'(t) = -2x_1 \end{cases} \qquad (E)$$

1. *Write (E) in matrix form* $X' = AX$.
2. *Is the origin an equilibrium point of system (E)?*
3. *Study the stability of system (E).*
4. *If* λ_1 *is a complex eigenvalue of* A *such that* $Im(\lambda_1) < 0$ *and* V_1 *is its associated eigenvector, determine the two functions:*

$$X_1(t) = Re(V_1 e^{\lambda_1 t}), \quad X_2(t) = Im(V_1 e^{\lambda_1 t})$$

5. *Find the general solution of system (E).*
6. *Deduce the solution of (E) with initial conditions* $x_1(0) = x_0$, $x_2(0) = y_0$.
7. *Using the definition, is the origin uniformly stable?*

Bibliography

[1] Abdelouahab, M. S. Les systèmes chaotiques à dérivées fractionnaires. (2009).

[2] Boyce, William E., and DiPrima, Richard C. *Elementary Differential Equations and Boundary Value Problems.* 10th ed. Wiley, 2012.

[3] Floquet, G. *Sur les équations différentielles linéaires à coefficients périodiques.* (1883).

[4] Goode, Stephen W. and Annin, Scott. *Differential Equations and Linear Algebra.* Upper Saddle River: Prentice Hall, 2000.

[5] Gourdon, X. *Les maths en tête, Analyse.* 3ème édition. Ellipses, 2020.

[6] Hadamard, Jacques. *Sur les problèmes aux dérivées partielles et leur signification physique.* Princeton University Bulletin, pages 49–52, 1902.

[7] H.K.Khalil, Nonlinear systems (third edition), Prentice Hall, 2002.

[8] LaSalle, J. P. *Stability Theory for Ordinary Differential Equations.* Journal of Differential Equations 4, 1968.

[9] Queffélec, H. and Zuily, C. *Analyse pour l'agrégation.* 3ème édition. Dunod, 2008.

[10] Ronga, F. *Analyse II.* Université de Genève, 2002. Disponible à l'adresse: `http://www.unige.ch/math/folks/ronga/lyse_II/2001-2002/chap_III.pdf`

[11] Rouvière, F. *Petit guide de calcul différentiel.* 2ème édition. Cassini, 2003.

[12] Strang, G. *Linear Algebra and Its Applications*. 5th ed. Boston, MA, USA: Cengage Learning, 2016.

[13] Zuily, C. *Uniqueness and Non-Uniqueness in the Cauchy Problem*. Vol. 33. Springer, 1983.